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SMOOTHING THEORY AND TOPOLOGY OF PL/O

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Abstract

The work of Thom, Milnor, Hirsch and Mazur has reduced the problems of smoothing theory to homotopy theory and study of the fibration of classifying spaces $PL/O \to BO \to BPL$.

In this paper, we give a detailed description of PL/O in low dimensions. The idea is that in a CW-approximation of a space X, the attaching maps (at least in the stable range) should be determined by the algebraic structure of $\pi_*(X)$ — π_*^S -module structure, Toda brackets etc.

In the case of PL/O, this algebraic structure is studied using "functorial" short exact sequences involving $\pi_n(PL/O)$ — especially the sequence of Kervaire and Milnor,

$$0 \to bP_{n+1} \to \Pi_n(PL/O) \to \Pi_n^S/ImJ \quad .$$

Thus we obtain a CW-approximation of PL/O in dimensions ≤ 10, and a Postnikov resolution in the same range.

Also is included a survey of the important developments in smoothing theory, and some general calculations of $\Gamma(M)$ (which are actually valid for any homotopy functor Γ) .

Ch. 1. Survey of smoothing theory.

1.1. Introduction and notation.

All manifolds will be Hausdorff and second countable, hence metrizable. Differentiable manifolds are $\mathscr{C}^\infty$-differentiable.

A PL-manifold is a manifold where all coordinate transformations are PL-isomorphisms. PL-manifolds may be triangulated as simplicial complexes K, where $\mathrm{link}(v,K)$ is PL-isomorphic to the standard PL-sphere or disc, for all vertices v in K. This may alternatively be taken as the definition of a PL-manifold (Hudson [13]).

To compare differentiable and PL-manifolds, we need the following definition:

A piecewise differentiable (PD-)homeomorphism f from a simplicial complex K onto a differentiable manifold M, is a homeomorphism such that $f|\sigma$ is differentiable of maximal rank ($= \dim \sigma$) for all simplexes σ of a triangulation of K. Then K is a PL-manifold, and we call f a differentiable triangulation.

Let K be a PL-manifold with PL-atals $\{(V_i,\psi_i)\}_{i\in I}$. A differentiable structure on K with atlas $\{(U_j,\varphi_j)\}_{j\in J}$ is said to be compatible with the PL-structure if each $\varphi_j\psi_i^{-1}:\psi_i(U_j\cap V_i)\to\varphi_j(U_j\cap V_i)$ is a PD-homeomorphism.

Then it is not difficult to see that we can identify a compatible differentiable structure on K with a pair (M,f), where $f : K\to M$ is a differentiable triangulation of M. In the following, a compatible differentiable structure will be called a smoothing.

Two smoothings (M,f) and (N,g) on K are equivalent if $g \circ f^{-1}$ is a diffeomorphism, and we let $\mathcal{S}(K)$ be the set of equivalence-classes. (This means that we identify structures with the same maximal atlas).

Now let

$\mathcal{PL}$ = PL-isomorphism classes of PL-manifolds

$\mathcal{Diff}$ = diffeomorphism classes of differentiable manifolds.

(both without boundary).

From Whitehead's triangulation theorems [37] we get a well-defined function $\mathcal{T}: \mathcal{Diff} \to \mathcal{PL}$, such that $K \in \mathcal{T}[M]$ if and only if there exists a differentiable triangulation $f : K \to M$.

The study of the function $\mathcal{T}$ has been one of the main problems in the topology of manifolds - starting with Milnor's famous paper [21] where he constructed several non-diffeomorphic smoothings on S^7, with the standard triangulation.

The next important step was taken by Thom, who showed the existence of PL 8-manifolds without any compatible differentiable structure ([34], see also [29]). Kervaire even constructed a PL 10-manifold without any differentiable manifold in its homotopy type [17].

So we divide the study of $\mathcal{T}$ in two parts:

1) Existence: If K is a PL-manifold, is $[K] \in$ in $\mathcal{T}$?

2) Classification: If $[K] \in$ in $\mathcal{T}$, what is $\mathcal{T}^{-1}([K])$?

$$\mathcal{T}^{-1}([K]) = \text{the set of diffeomorphism classes of smoothing on } K$$
$$= \mathcal{S}(K)/\sim ,$$

where $(M,f) \sim (N,g)$ if and only if M and N are diffeomorphic. We will use the notation $\mathcal{D}(K) = \mathcal{T}^{-1}([K])$.

This classification problem is in general very difficult, and we therefore will introduce a new set, $\Gamma(K)$, which in some sense is an approximation to $\mathscr{D}(K)$.

Two smoothings (M_1, f_1) and (M_2, f_2) are called <u>concordant</u> if there is a smoothing (W, F) on $K \times I$ and diffeomorphisms $g_i : M_i \to \partial W_i = F(K \times (i-1))$, $i = 1,2$, such that the diagram

$$\begin{array}{ccccccc}
M_1 & \xrightarrow{g_1} & \partial W_1 \subset W \supset \partial W_2 & \xleftarrow{g_2} & M_2 \\
{\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle F} & & \uparrow{\scriptstyle f_2} \\
K \times 0 & \subset & K \times I & \supset & K \times 1
\end{array}$$

commutes.

It is easy to see that this is an equivalence relation, and we let the set of equivalence classes be $\Gamma(K)$.

Another relation is the following: Two smoothings (M_1, f_1) and (M_2, f_2) of K are called <u>isotopic</u> if there exists a PD isotopy $h : K \times I \to M_1 \times I$ (a PD homeomorphism (triangulation) preserving the second coordinate) such that $h_0 = f_1$ and (M_2, f_2) is equivalent to (M_1, h_1) ($f_2 \circ h_1^{-1}$ is a diffeomorphism). This is also an equivalence relation (see [10]), and the set of equivalence classes we call $I(K)$.

It is clear that isotopy is a stronger relation than both concordance and diffeomorphism, so we have the diagram

$$\begin{tikzcd}[row sep=tiny]
 & & \Gamma(K) \\
\mathscr{S}(K) \arrow[r] & I(K) \arrow[ur] \arrow[dr] & \\
 & & \mathscr{D}(K)
\end{tikzcd}$$

where all maps are surjective.

The relation between concordance and isotopy is given in the following theorem of Hirsch [10].

Theorem. Let (W,F) be a smoothing of $K \times I$, inducing a smoothing (M,f) on $K \times 0$. Then the smoothings (W,F) and $(M \times I, f \times id)$ of $K \times I$ are isotopic by an isotopy leaving $K \times 0$ fixed.

(Weaker versions may be found in [26] and [27]).

Corollary. The natural map $I(K) \to \Gamma(K)$ is a bijection.

Hence we get a surjective map $\Gamma(K) \to \mathcal{D}(K)$.

From now on we will concentrate on the set $\Gamma(K)$, and therefore the "classification problem" in the following will mean: Given $K \in$ in $\mathcal{T}$, calculate $\Gamma(K)$.

1.2. Obstruction theories and homotopy classification of smoothings.

Thom was the first to look at these problems as obstruction problems (see [35]). Then Munkres constructed obstruction theories of this type ([24] and [25]), but his methods were complicated and the results somewhat unsatisfactory, since he was not able to prove that the differentiable structures he obtained were compatible (He has proved this later).

The most elegant and satisfactory of these "geometric" obstruction theories was outlined by Hirsch in [9]. His theories depend on the following "Product theorem", proved by Hirsch [8] and Cairns [6].

Theorem. The natural map $\Gamma(K) \to \Gamma(K \times \mathbb{R}^n)$, (product with the standard structure on $\mathbb{R}^n$) is a bijection.

The obstruction theories all take values in cohomology with coefficients

Γ_i = oriented diffeomorphism classes of smoothings of S^i (Hirsch)

$\approx$ diffeomorphisms of S^{i-1}, modulo those extendable to the whole disc B^i.

From the obstruction theories (or by a direct argument - see [18]) it also follows that $\Gamma_i \approx \Gamma(S^i)$.

The first step towards a homotopy theoretic formulation of the obstruction theories was taken by Milnor [22]. He introduced the concept of microbundles, and defined the tangent microbundle of a manifold (Diff., PL, or top.). (Since that, several people have shown (see [14]) that microbundles contain bundles in an essentially unique way, so we will only use the word "bundle").

Let $k_{PL}(K)$ = stable equivalence classes of PL-bundles over K and $k_O(K)$ = stable vectorbundles over K.

Then we may use Brown's representation theorem and construct classifying spaces BPL and BO for k_{PL} and k_O [30]. We may also construct a natural transformation $k_O \to k_{PL}$, and hence a continuous map $p : BO \to BPL$ (homotopy - unique on skeletons).

Milnor proved the following

<u>Theorem</u>. Let K be a PL-manifold with stable tangent bundle classified by $f : K \to BPL$. Then K has a smoothing if and only if f lifts in homotopy through p. That is - there exists $\tilde{f} : K \to BO$ such that $p \circ \tilde{f} \simeq f : M \to BPL$.

This theorem gives an obstruction theory with coefficients $\pi_i(BPL,BO)$, and Hirsch established the connection between the two types of obstruction theories - showing that $\pi_i(BPL,BO) \approx \Gamma_{i-1}$ [9].

Now we replace $p : BO \to BPL$ with a fibration, and call the fibre PL/O (the fibre has a well defined homotopy type). Since $k_{PL}(X)$ and $k_O(X)$ are groups with the addition defined by Whitney-sum of bundles, BPL and BO are H-spaces, and since $k_O(X) \to k_{PL}(X)$ is a group homomorphism, we may suppose that $p : BO \to BPL$ respects the H-space structures. BO and BPL are homotopy commutative and homotopy associative, and PL/O inherits an H-space structure with the same properties.

Obviously, $\pi_i(BPL,BO) \approx \pi_{i-1}(PL/O)$, and therefore we may reformulate Hirsch's result as $\Gamma_i \approx \pi_i(PL/O)$. A generalization of this result is the following beautiful classification theorem, due to Hirsch and Mazur ([11], see also [18]).

<u>Theorem.</u> Suppose K has a smoothing. Then there is a bijection

$\Gamma(K) \approx [K,PL/O]$, such that the given smoothing corresponds to the constant map $K \to PL/O$.

If $f : K \to BPL$ is a continuous map, we let
$\mathcal{L}(f) = \{$fibre homotopy classes of liftings of f through $p : BO \to BPL\}$.

The fibration $PL/O \to BO \to BPL$ is "principal" in the sense of homotopy theory, and therefore $\mathcal{L}(f) \approx [K,PL/O]$, provided we know that $\mathcal{L}(f) \neq \emptyset$.

Now let $f : K \to BPL$ be a classifying map for the stable tangent bundle of K. Milnor's existence theorem says that $\Gamma(K)$ is nonempty if and only if $\mathcal{L}(f)$ is nonempty, and therefore we can formulate both the existence and classification theorem in the following elegant way:

<u>Theorem</u>. Let $f : K \to BPL$ be classifying for the stable tangent

bundle of K. Then there is a bijection

$$\Gamma(K) \approx \mathcal{L}(f).$$

Immediate corollaries of these results are

1) Stably parallelizable PL manifolds are always smoothable.

2) If $\Gamma(K) \neq \emptyset$, it only depends on the homotopy type of K.

Here the condition $\Gamma(K) \neq \emptyset$ is necessary, because Hirsch and Milnor have constructed non-smoothable PL manifolds of the homotopy type of differentiable manifolds [12].

Given a smoothing of K, we see from the bijection $\Gamma(K) \approx [K,PL/O]$ that $\Gamma(K)$ has an abelian group structure, with the given smoothing as zero element.

$\Gamma(S^n)$ is also a group under the formation of connected sum, and if we take the given smoothing on S^n to be the standard smoothing, it can be shown that $\Gamma(S^n) \approx [S^n,PL/O]$ is an isomorphism of groups.

The fundamental work on Γ_n is the work of Kervaire and Milnor [15]. They study an other group - θ_n = h-cobordism of oriented homotopy spheres under connected sum, but it follows from the h-cobordism theorem that $\theta_n \approx \Gamma_n$ for $n \geq 5$. From their work we therefore get that Γ_n is finite for $\Gamma_n \geq 5$.

Cerf has shown that $\Gamma_4 = 0$ [7], it follows from Smale's work [31] that $\Gamma_3 = 0$, and it is well known that $\Gamma_i = 0$ for $i < 3$. Therefore Γ_n is finite for all n.

We should also mention the important work of Brumfiel, who has determined more exactly the algebraic structure of θ_n, and hence also of Γ_n [4]. For $n \leq 17$, we have the following table

n	≤ 6	7	8	9	10	11	12	13	14	15	16	17
Γ_n	0	Z_{28}	Z_2	$3Z_2$	Z_0	Z_{992}	0	Z_3	Z_2	$Z_{8128}+Z_2$	Z_2	$4Z_2$

Finally, some comments on $\mathcal{D}(K)$. From [15] and the isomorphism $\theta_n \approx \Gamma_n = \{$oriented diffeomorphism classes of (oriented) differentiable structures on $S^n\}$, it follows that if Σ^n is an element in Γ_n, we get $-\Sigma^n$ just by changing the orientation. It is then easy to see that $\mathcal{D}(S^n) \approx \Gamma_n/(\Sigma^n \sim -\Sigma^n)$. We therefore may read off $\mathcal{D}(S^n)$ as soon as we know the algebraic structure of Γ_n.

For example, we get from the table above that $|\mathcal{D}(S^7)| = 15$ and $|\mathcal{D}(S^8)| = 2$.

In general, $\Gamma_n \approx \mathcal{D}(S^n)$ if and only if all elements in Γ_n are of order two. Otherwise $|\Gamma_n| > |\mathcal{D}(S^n|$.

Kervaire and Milnor have shown [15] that all homotopy spheres are stably parallelizable, and therefore two homotopy spheres of the same dimension are (stably) tangentially homotopy equivalent. But then it follows from Mazur's stable isotopy theorem [20] and the fact that $|\mathcal{D}(S^n)| < \infty$, that $|\mathcal{D}(S^n \times \mathbb{R}^m)| = 1$ for m large.

From this we get

<u>Corollary</u>. There is no "classifying space" for $\mathcal{D}$. This means, there is no topological space X, such that $\mathcal{D}(K) \approx [K,X]$ for all K with $\mathcal{D}(K) \neq \emptyset$.

<u>Proof</u>. Suppose X existed. Then we would have

$\mathcal{D}(K) \approx [K,X] \approx [K \times \mathbb{R}^n, X] \approx \mathcal{D}(K \times \mathbb{R}^n)$, which contradicts the result above.

Before leaving $\mathcal{D}(K)$, we just mention that since $\Gamma(K)$ is finite for K of the homotopy type of a finite complex (see chapter 2), the same is true for $\mathcal{D}(K)$ - because of the surjection $\Gamma(K) \to \mathcal{D}(K)$.

Ch. 2. Computation of $\Gamma(M)$ in some special cases.

For notational convenience we now will define $\Gamma(X) = [X, PL/O]$ for any CW-complex X.

First we remark that $\Gamma(M) = 0$ for M contractible. Secondly, since PL/O is 6-connected $(\pi_i(PL/O) = 0$ for $i \leq 6)$, $\Gamma(M) = 0$ for $\dim M \leq 6$.

If M is a closed, connected 7-manifold, we may write $M = K \cup_f e^7$ where K is a complex of dimension ≤ 6 and $f : S^6 \to K$ is a continuous map. The Puppe sequence

$$S^6 \xrightarrow{f} K \to M \to S^7$$

then gives exact sequence of abelian groups

$$\Gamma(S^7) \to \Gamma(M) \to (K) \to \Gamma(S^6).$$

$\Gamma(K) = 0$ since $\dim K \leq 6$, hence $\Gamma(M)$ is a quotient of $\Gamma(S^7) \approx \Gamma_7 \approx \mathbb{Z}_{28}$, and $|\Gamma(M)| \leq 28$. Later (Ch. 5) we shall see that $\Gamma(M^7) \approx Z_{28}$ if M^7 is orientable, and $\Gamma(M^7) \approx Z_2$ if M^7 is non-orientable.

In general we cannot find an upper limit for $|\Gamma(M)|$, but at least we have

Theorem. If M has the homotopy type of a finite complex, then $\Gamma(M)$ is finite.

Proof. $\pi_i(PL/O)$ is finite for all i.

Now we shall study particular classes of manifolds, and it is natural to start with products of spheres.

Theorem. $\Gamma(S^p \times S^q) \approx \Gamma_p \oplus \Gamma_q \oplus \Gamma_{p+q}$. (Mazur)

Proof. We use the Puppe sequence

$$S^p \vee S^q \xrightarrow{\iota} S^p \times S^q \xrightarrow{\pi} S^p \wedge S^q = S^{p+q} \to S(S^p \vee S^q) \xrightarrow{S\iota} S(S^p \times S^q).$$

which gives exact sequence of abelian groups

$$\Gamma(S(S^p \times S^q)) \xrightarrow{(S\iota)^*} \Gamma(S(S^p \vee S^p)) \to \Gamma(S^{p+q}) \to \Gamma(S^p \times S^q) \xrightarrow{\iota^*} \Gamma(S^p \vee S^q)$$

$S\iota$ has a left homotopy inverse, defined by

$$S(S^p \times S^q) \to S(S^p \times S^q) \vee S(S^p \times S^q) \xrightarrow{S(pr_1) \vee S(pr_2)} S(S^p) \vee S(s^q) \simeq S(S^p \vee S^q).$$

Thus $(S\iota)^*$ is surjective, and we get exact sequence (note that $\Gamma(S^p \vee S^q) \approx \Gamma(S^p) \oplus \Gamma(S^q)$)

$$0 \to \Gamma_{p+q} \to \Gamma(S^p \times S^q) \xrightarrow{\iota^*} \Gamma_p \oplus \Gamma_q$$

But ι^* has the following right inverse

$$\rho : [S^p, PL/O] \times [S^q, PL/O] \to [S^p \times S^q, PL/O \times PL/O] \to [S^p \times S^q, PL/O],$$

where the second map is induced by the multiplication on PL/O. Therefore ι^* is split surjective, and the theorem follows.

Both ι^* and ρ in this proof have easy geometric interpretations as maps between $\Gamma(S^p \times S^q)$ and $\Gamma_p \times \Gamma_q$.

ρ defines the product structure on $S^p \times S^q$.

Conversely, given a smoothing on $S^p \times S^q$, we get induced smoothings on product neighborhoods of $S^p \subset S^p \times S^q$ and $S^q \subset S^p \times S^q$. The Cairns-Hirsch product theorem then gives elements in Γ_p and Γ_q.

These constructions we may of course do for any pair of manifolds M_1 and M_2, and we actually have

Theorem $\Gamma(M_1 \times M_2) \approx \Gamma(M_1) \oplus \Gamma(M_2) \oplus \Gamma(M_1 \wedge M_2)$.

For example, this means that $|\Gamma(M_1 \times M_2)|$ is divisible by $|\Gamma(M_1)|\cdot|\Gamma(M_2)|$.

The last summand has no natural geometric interpretation in general, but in some cases it may be calculated.

For example, $S^p \wedge (S^q \times S^r) \simeq S^{p+q} \vee S^{p+r} \vee S^{p+q+r}$ (See proof of theorem above), and

$$\Gamma(S^p \wedge (S^q \times S^r)) \approx \Gamma_{p+q} \oplus \Gamma_{p+r} \oplus \Gamma_{p+q+r}$$

Using this, the following is easily proved by induction.

<u>Theorem</u>. $\Gamma(S^{p_1} \times \ldots \times S^{p_n}) \approx \sum_{1 \leq i \leq n} \Gamma_{p_i} \oplus \sum_{1 \leq i < j \leq n} \Gamma_{p_i+p_j} \oplus \ldots \oplus \Gamma_{p_1+\ldots+p_n}$.

For the torus $T^n = (S^1)^n$ we get

<u>Corollary</u> $$\Gamma(T^n) \approx \sum_{m=7}^{n} \binom{n}{m} \Gamma_m$$

(where $k\Gamma_m = \Gamma_m \oplus \ldots \oplus \Gamma_m$, k copies).

Next we study sphere bundles over spheres, that is, we want to calculate $\Gamma(E)$, where E is the total space of an orthogonal sphere bundle $S^p \xrightarrow{i} E \xrightarrow{\pi} S^q$.

Let $g : S^{q-1} \to O(p+1)$ ($g : S^{q-1} \to SO(p+1)$ if $q > 1$) be a characteristic map for this bundle. Then the homotopy type of E (and hence $\Gamma(E)$) is only depending on $[g] \in \pi_{q-1}(O(p+1))$.

Suppose now that π has a section. Then g lifts (in homotopy) to $\bar{g} : S^{q-1} \to O(p)$. For $m \geq p$, the inclusion $O(p) \to O(m)$ and the J-homomorphism define a homomorphism

$$J_m : \pi_{q-1}(O(p)) \to \pi_{q-1}(O(m)) \xrightarrow{J} \pi_{m+q-1}(S^m)$$

Define $\tau_{\bar{g}} : \pi_m(PL/O) \to \pi_{m+q-1}(PL/O)$ to be the homomorphism induced by $J_m(\bar{g}) : S^{m+q-1} \to S^m$.

<u>Theorem.</u> Let $S^p \to E \to S^q$ be an orthogonal bundle admitting a cross-section, with characteristic map $g : S^{q-1} \to O(p)$. Then we have short-exact sequence

$$0 \to \Gamma_{p+q}/\mathrm{im}\ \tau_g \to \Gamma(E) \to \Gamma_q \oplus (\Gamma_p \cap \ker \tau g) \to 0.$$

(A slightly different version is found in [27]).

<u>Proof.</u> We may write E in the following way:

$$E = S^p \times e^q/((u,v) = (g(v)u,v_0) \quad \text{for all} \quad v \in \partial e^q)$$

($v_0 \in \partial e^q$ is a base point). Since $g(v) \in SO(p)$, there exists $u_0 \in S^p$ such that $g(v)u_0 = u_0$ for all v. Therefore we have (up to homotopy)

$$E = S^p \times v_0 \cup u_0 \times S^q \cup_f e^p \times e^q, \quad \text{where}$$

$$f(u,v) = \begin{cases} (u_0,v) & \text{if} \quad u \in \partial e^p \\ (g(v)u,v_0) & \text{if} \quad v \in \partial e^q \end{cases}$$

with the natural identifications $e^q/\partial e^q = S^q$ and $e^p/\partial e^p = S^p$ compatible with the $SO(p)$-operation.

Then we have the Puppe sequence

$$S^{p+q-1} \xrightarrow{f} S^p \vee S^q \to E \to S^{p+q} \xrightarrow{Sf} S^{p+1} \vee S^{q+1} \to SE \to \ldots$$

which gives exact sequence of abelian groups

$$\ldots \to \Gamma_{p+1} \oplus \Gamma_{q+1} \xrightarrow{(Sf)^*} \Gamma_{p+q} \to \Gamma(E) \to \Gamma_p \oplus \Gamma_q \xrightarrow{f^*} \Gamma_{p+q-1}$$

and hence a short exact sequence

$$0 \to \Gamma_{p+q}/\mathrm{im}(Sf)^* \to \Gamma(E) \to \ker f^* \to 0 .$$

It remains to identify the homomorphisms f^* and $(Sf)^*$.

If $\sigma : S^q \to E$ is a section in the bundle, the composite homomorphism $\Gamma(E) \to \Gamma_p \oplus \Gamma_q \xrightarrow{pr} \Gamma_q$ is equal to $\sigma^* : \Gamma(E) \to \Gamma_q$, which is (split) surjective. Therefore f^*/Γ_q is zero, and we may identify f^* with

$$\Gamma_q \oplus \Gamma_p \xrightarrow{pr} \Gamma_p \xrightarrow{\bar{f}^*} \Gamma_{p+q-1} \text{ , where}$$

$\bar{f} : S^{p+q-1} \xrightarrow{f} S^p \vee S^q \xrightarrow{pr} S^p$. But $\bar{f}$ is then up to sign a representative for $J(g) \in \pi_{p+q-1}(S^p)$, such that $\bar{f}^* = \pm \tau_g : \Gamma_p \to \Gamma_{p+q-1}$.

In the same manner we see that $(Sf)^*/\Gamma_{q+1}$ is zero, and $\overline{Sf} = S\bar{f}$ represents $\pm SJ(g) \in \pi_{p+q}(S^{p+1})$. From the commutative diagram

$$\begin{array}{ccc}
\pi_{q-1}(O(p)) & \longrightarrow & \pi_{q-1}(O(p+1)) \\
\downarrow J & & \downarrow J \\
\pi_{p+q-1}(S^p) & \xrightarrow{S} & \pi_{p+q}(S^{p+1})
\end{array}$$

it then follows that

$$(\overline{Sf})^* = \pm \tau g : \Gamma_{p+1} \to \Gamma_{p+q} .$$

Thus $\ker f^* = \Gamma_q \oplus (\Gamma_p \cap \ker \tau_g)$ and $\operatorname{im}(Sf)^* = \operatorname{im} \tau_g \cap \Gamma_{p+q}$.

<u>Remark</u>. If the bundle is trivial, then obviously $\tau_g = 0$, and we get the exact sequence

$$0 \to \Gamma_{p+q} \to \Gamma(S^p \times S^q) \to \Gamma_p \oplus \Gamma_q \to 0$$

which is easily proved to split.

In general it is clear that Γ_q splits off as a direct summand in $\Gamma(E)$, but I do not know if the entire sequence splits. In any case we get, since all groups are finite:

$$|\Gamma(E)| = |\Gamma_q| \cdot |\Gamma_p \cap \ker \tau_g| \cdot |\Gamma_{p+q}/\mathrm{im}\ \tau_g|$$

The easiest example of a bundle with $\tau_g \neq 0$, is perhaps the non-trivial 8-sphere bundle over S^2. In this case $\mathrm{Im}(g) =$ generator (= the stable Hopf map η) in both $\pi_8(S^8)$ and $\pi_{10}(S^9)$, and it will follow from the calculations in chapter 4 that $\tau_g = \eta^* : \Gamma_8 \to \Gamma_9$ is injective, and that coker $[\tau_g : \Gamma_q \to \Gamma_{10}] \approx Z_3$. Since $\Gamma_2 = 0$, we therefore get $\Gamma(E) \approx \mathbf{Z}_3$.

Even if the bundle does not admit any cross-section, it is possible to establish a similar result, using the result that PL/O is a loop-space. I will only sketch the proof.

We look at the Puppe sequence

$$S^p \xrightarrow{\iota} E \to E \cup_\iota e^{p+1} \xrightarrow{h} S^{p+1} \to SE ,$$

where ι is inclusion of a fibre. Up to homotopy we may identify $E \cup_\iota e^{p+1}$ with $S^{p+q} \vee S^q$, and h with a map which on S^{p+q} is homotopic to the map inducing τ_g, and an S^q is the suspension of $\bar{g} : S^{q-1} \xrightarrow{g} O(p+1) \xrightarrow{ev} S^p$, where $ev(A) = A(u_o)$, $u_o \in S^p$.

We now get exact sequence

$$\cdots \to \Gamma_{p+1} \xrightarrow{(S\bar{g})^* \oplus \tau_g} \Gamma_q \oplus \Gamma_{p+q} \to \Gamma(E) \to \Gamma_p \qquad (*)$$

If we write $PL/O \simeq \Omega X$, we may continue this sequence to the right:

$$\begin{array}{ccccc}
\Gamma(E) & \longrightarrow & \Gamma_p & \xrightarrow{\rho} & \Gamma_{q-1} \oplus \Gamma_{p+q-1} \\
\| & & \| & & \wr\wr \\
[E,PL/O] & \to & [S^p,PL/O] & & [S^{q-1} \vee S^{p+q-1}, PL/O] \\
\wr\wr & & \wr\wr & & \wr\wr \\
[SE,X] & \to & [S^{p+1},X] & \xrightarrow{h^*} & [S^q \vee S^{p+q}, X]
\end{array}$$

Thus we have

<u>Theorem</u>. There is an exact sequence

$$0 \to \Gamma_q/\mathrm{im}(S_{\bar{g}})^* \oplus \Gamma_{p+q}/\mathrm{im}\ \tau_g \to \Gamma(E) \to \mathrm{Ker}\ \rho \to 0$$

where ρ is a homomorphism $\rho : \Gamma_p \to \Gamma_{q-1} \oplus \Gamma_{p+q-1}$.

Ch. 3. Exact sequences involving Γ_n.

Two well-known exact sequences are deduced and given a "functorial" form, which will be useful for the calculations in the next chapter.

First, look at the long exact homotopy sequence of the fibration $PL/O \xrightarrow{i} BO \xrightarrow{p} BPL$

$$\cdots \to \pi_n(BO) \xrightarrow{p_*} \pi_n(BPL) \xrightarrow{\delta} \pi_{n-1}(PL/O) \xrightarrow{i_*} \pi_{n-1}(BO) \to \cdots$$

<u>Theorem</u>. (Hirsch-Mazur) With the identification

$$\Gamma_k = \pi_k(PL/O) \text{ , the sequence}$$

$$0 \to \pi_n(BO) \xrightarrow{p_*} \pi_n(BPL) \to \Gamma_{n-1} \to 0$$

is exact (for all n).

<u>Proof</u>. It is sufficient to prove that $p_* : \pi_n(BO) \to \pi_n(BPL)$ is injective for all n.

(i) $n \equiv 0 \bmod 4$. Then $\pi_n(BO) \approx \mathbf{Z}$, and since Γ_n is finite, $i_* : \pi_n(PL/O) \to \pi_n(BO)$ must be zero.

(ii) $n \equiv 3,5,6,7 \bmod 8$. Then $\pi_n(BO) = 0$, and p_* is trivially injective.

(iii) $n = 1,2 \bmod 8$, $\pi_n(BO) \approx Z_2$.

The diagram

$$\begin{tikzcd}
\pi_n(BO) \arrow[r, "p_*"] \arrow[d, phantom, "\wr"] & \pi_n(BPL) \arrow[r] & \pi_n(BF) \arrow[d, phantom, "\wr"] \\
\pi_{n-1}(O) \arrow[rr] \arrow[drr, "J"'] & & \pi_{n-1}(F) \arrow[d, phantom, "\wr"] \\
 & & \pi^s_{n-1}
\end{tikzcd}$$

commutes, where $F = \varinjlim_{n} F_m$, F_m = homotopy equivalences of S^m , BF = classifying space for (stable) sphere fibrations, and all homomorphisms are the natural ones. But Adams has shown [1] that J is injective in these dimensions, and therefore also p_* is injective.

We may suppose that the map $q : BPL \to BF$ is a fibration, and we call the fibre F/PL. Then the composition $q \circ p : BO \to BF$ is also a fibration, and we call the fibre F/O. The following diagram is commutative

$$\begin{array}{ccccc}
PL/O & \xrightarrow{\iota} & F/O & \xrightarrow{\pi} & F/PL \\
\downarrow & & \downarrow & & \downarrow \\
BO & = & BO & \xrightarrow{p} & BPL \\
p\downarrow & & \downarrow q\circ p & & \downarrow q \\
BPL & \xrightarrow[q]{} & BF & = & BF
\end{array}$$

and it is easy to see that $PL/O \xrightarrow{\iota} F/O \xrightarrow{\pi} F/PL$ also is a fibration.

Now, look at the following diagram

$$\begin{array}{ccccccccc}
0 \to & \pi_{n+1}(BO) & \to & \pi_{n+1}(BPL) & \to & \pi_n(PL/O) & \to 0 & & \\
 & \| & & \downarrow & & \downarrow & & & \\
\ldots \to & \pi_{n+1}(BO) & \to & \pi_{n+1}(BF) & \to & \pi_n(F/O) & \to \ldots & & \\
 & \wr & & \wr & & \cup & & & \\
 & \pi_n(O) & \xrightarrow{J} & \pi_n(F) & \to & \pi_n(F)/\mathrm{im}\, J & \approx & \pi_n^s/\mathrm{im}\, J &
\end{array}$$

The upper row is exact by the theorem above (Hirsch-Mazur), and the middle row is part of the long exact homotopy sequence of the fibration $F/O \to BO \to BF$. From this diagram we see that

$$\operatorname{im} \iota_* \subset \pi_n^s / \operatorname{im} J \subset \pi_n(F/O)$$

so from the homotopy sequence of $PL/O \to F/O \to F/PL$, we get an exact sequence

$$\pi_n(PL/O) \xrightarrow{\iota_*} \pi_n^s / \operatorname{im} J \xrightarrow{\alpha} \pi_n(F/PL) ,$$

where $\alpha = \pi_* | \pi_n^s/\operatorname{im} J : \pi_n^s/\operatorname{im} J \subset \pi_n(F/O) \to \pi_n(F/PL)$.

$\pi_*(F/PL)$ has been calculated by Sullivan [34]:

$\pi_{4k}(F/PL) \approx \mathbb{Z}$, $\pi_{4k+2}(F/PL) \approx \mathbb{Z}_2$, and $\pi_{2k+2}(F/PL) \approx 0$.

Therefore $\bar{\iota}_*$ must be surjective for $n \not\equiv 2 \bmod 4$ (since π_n^s is finite, $n > 0$).

For $n = 4k + 2$ we may identify the homomorphism

$$\begin{array}{ccc} \pi_n^s & \xrightarrow{\beta} & \pi_n(F/PL) \\ \wr\wr & & \wr\wr \\ \Omega_n^{fr} & \longrightarrow & \mathbb{Z}_2 \end{array}$$

with the Kervaire invariant, which is zero for $n \neq 2^m-2$ [2]. Hence $\bar{\iota}_*$ is surjective for $n \neq 2^m-2$.

Inspecting the cobordism interpretation of $\pi_{n+1}(F/PL)$ [34] and the connecting homomorphism $\delta : \pi_{n+1}(F/PL) \to \pi_n(PL/O)$, we may show that $\operatorname{im} \delta = \ker \iota_* = \ker \bar{\iota}_* = bP_{n+1}$ in Kervaire-Milnor [15] ($n \geq 5$), and therefore we have

<u>Theorem</u>. (Kervaire-Milnor [15]). There is an exact sequence

$$0 \to bP_{n+1} \to \Gamma_n \to \pi_n^s/\operatorname{im} J$$

for $n \geq 5$. Moreover, the right-hand homomorphism is surjective if $n \neq 2^m-2$.

Since bP_{n+1} is the image of $\pi_{n+1}(F/PL)$, it is cyclic - zero if n even, and at most of order 2 if $n \equiv 1 \bmod 4$.

Remark. It follows from recent work of May, Madsen and Milgram [19] that $\operatorname{coker}(\Gamma_n \to \pi_n^s/\mathrm{im}\, J) \approx \mathbf{Z}_2$ for $n = 2^m-2$. Together with the solution of the Adams conjecture and Brumfiel's splitting theorems, this completely determines Γ_n.

The reason why we introduce Kervaire-Milnor's exact sequence this way, is that it now clearly possesses the following "functorial" property:

Let $\varphi : S^{n+k} \to S^n$ be a continuous map. φ induces homomorphisms $\varphi^* : \pi_n(PL/O) \to \pi_{n+k}(PL/O)$ and $\varphi^* : \pi_n^s/\mathrm{im}\, J \to \pi_{n+k}^s/\mathrm{im}\, J$, hence also φ^*: $\operatorname{Ker} \iota_{*n} \to \operatorname{Ker} \iota_{*n+k}$. Then the following diagram commutes:

$$\begin{array}{ccccccc}
0 & \longrightarrow & bP_{n+1} & \longrightarrow & \Gamma_n & \longrightarrow & \pi_n^s/\mathrm{im}\, J \\
 & & \downarrow{\varphi^*} & & \downarrow{\varphi^*} & & \downarrow{\varphi^*} \\
0 & \longrightarrow & bP_{n+k+1} & \longrightarrow & \Gamma_{n+k} & \longrightarrow & \pi_{n+k}^s/\mathrm{im}\, J
\end{array}$$

This property will be of fundamental importance for the calculations in chapter 4.

Ch. 4. Calculations in $H^*(PL/O)$.

The idea is to use the exact sequences in Ch. 3 to obtain information on the global homotopy structure of PL/O (i.e. π_*-module structure, Toda brackets etc. in $\pi_* PL/O$). This is used to construct a CW-approximation of PL/O.

In this chapter, theorems and tables in Toda: "Composition methods in Homotopy groups of Spheres" (referred to as TODA) will play an important role.

4.1. Localization of H-spaces.

If G is an abelian group and p a prime, we let $G_{(p)} = G \otimes \mathbb{Z}_{(p)} = G$ localized at p.

Let X be a homotopy commutative H-space. Then the functor $[-,X]$ takes values in the category of abelian groups, and we can construct the functor $[-,X]_{(p)}$. Since localization preserves exactness, it is easy to see that $[-,X]_{(p)}$ satisfies the conditions of Brown's representation theorem [3]. Hence we get a new homotopy commutative H-space $X_{(p)}$ such that $[-,X]_{(p)} \approx [-,X_{(p)}]$ on the category of CW-complexes, and - if X is a CW-complex - a mapping $X \to X_{(p)}$, representing the localization

$$[-,X] \to [-,X]_{(p)} \approx [-,X_{(p)}].$$

$X_{(p)}$ is well defined up to homotopy, and we call it "X localized at p".

Suppose now that $\pi_i(X)$ is finite for all i. Then $\pi_i(X) \to \pi_i(X_{(p)})$ is the projection onto the p-primary component of $\pi_i(X)$.

Now we may localize at all primes p, and form the product and the map

$$\phi : X \longrightarrow \prod_p X_{(p)} .$$

Then $$\phi_* : \pi_*(X) \to \pi_*(\prod_p X_{(p)}) \approx \prod_p(\pi_*(X)_{(p)})$$

is an isomorphism, and therefore ϕ is a weak homotopy equivalence.

Let F_{cp} be the class of finite groups of order prime to p. Since $\pi_*(X) \to \pi_*(X_{(p)})$ is an isomorphism mod F_{cp}, $H_*(X) \to H_*(X_{(p)})$ will also be an isomorphism mod F_{cp}. But since $H_*(X) \approx H_*(X)_{(p)}$ mod F_{cp}, we get $H_*(X_{(p)}) \approx H_*(X)_{(p)}$ mod F_{cp}.

Next, let F_p be the class of p-primary finite groups. Then $\pi_*(X_{(p)}) = 0$ mod F_p, and so $H_*(X_{(p)}) = 0$ mod F_p. On the other hand, $\tilde{H}_*(X)_{(p)} = 0$ mod F_p since $H_i(X)$ is finite for all $i > 0$.

From all this we get

$$\tilde{H}_*(X_{(p)}) \approx \tilde{H}_*(X)_{(p)} = p\text{-torsion } (\tilde{H}_*(X))$$

and hence, since $\tilde{H}_*(X) = \sum_p \tilde{H}_*(X)_{(p)}$

$$\tilde{H}_*(X) \approx \sum_p \tilde{H}_*(X_{(p)}).$$

Therefore: to calculate $H_*(X)$ - and hence $H^*(X)$ - it suffices to calculate $H_*(X_{(p)})$ for all primes p. Furthermore, it follows easily that $H^*(X_{(p)} : \mathbf{Z}_p) \to H^*(X : \mathbf{Z}_p)$ is an isomorphism of modules over the Steenrod algebra.

To show that $\tilde{H}^*(X) \approx \prod_p \tilde{H}^*(X_{(p)})$ multiplicatively; let $px = qy = 0$, with $(p,q) = 1$, and $x,y \in \tilde{H}^*(X)$. Then we can find r and s such that $rp + sq = 1$, and therefore

$$x \cup y = (rp+sq)x \cup y = r(px) \cup y + sx \cup (qy) = 0 .$$

All this now applies to PL/O, since $\pi_i PL/O$ is finite for all i. To study $\pi_i PL/O_{(p)} \approx \Gamma_{i(p)}$, we use the localized versions of the sequences in ch. 3, which exist since localization preserves exactness.

4.2. $H^*(PL/O_{(p)})$ for p odd.

We will only study PL/O in the stable range, and there we have the following table of $\pi_i(PL/O)$:

i	≤ 6	7	8	9	10	11	12	13
$\pi_i(PL/O)$	0	$Z_4 \oplus Z_7$	Z_2	$3Z_2$	$Z_2 \oplus Z_3$	$Z_{32} \oplus Z_{31}$	0	Z_3

From this we see that in this range $(* \leq 13)$ we have

$$H^*(PL/O_{(7)}) \approx H^*(K(Z_7,7)) \quad \text{and} \quad H^*(PL/O_{(31)}) \approx H^*(K(Z_{31},11)).$$

The only odd prime left is 3, and $H^*(PL/O_{(3)})$ has been calculated by Williamson [38]. Since his method illustrates - and actually gave the idea to - the technique used in our calculations, we include it here.

In this range we have the Postnikov system

$$\begin{array}{ccc} K(Z_3,13) \longrightarrow & PL/O_{(3)} & \\ & \downarrow & \\ & K(Z_3,10) & \xrightarrow{k} K(Z_3,14) \end{array},$$

and we must determine $k \in H^{14}(K(Z_3,10),Z_3)$. This group is isomorphic to Z_3, generated by P_3^1, and since P_3^1 and $-P_3^1$ give equivalent fibrations, it is sufficient to decide whether $k = 0$ or $k \neq 0$.

Now we have $\pi_{10}(SO_{(3)}) \approx \pi_{13}(SO_{(3)}) \approx 0$, so $\mathrm{Im}\, J_{(3)} = 0$

in these dimensions. Furthermore, $bP_{11(3)} \approx bP_{14(3)} \approx 0$ [15], and we have functorial isomorphisms (ch. 3) :

$$\pi_{10}(PL/O_{(3)}) \approx \pi^s_{10(3)} \quad \text{and}$$

$$\pi_{13}(PL/O_{(3)}) \approx \pi^s_{13(3)} .$$

Let now $\alpha_1 \in \pi^s_{3(3)} \approx Z_3$ be the generator.

α_1 induces the diagram

$$\begin{array}{ccc} \pi_{10}(PL/O_{(3)}) & \approx & \pi^s_{10(3)} \\ \alpha_1{}^*\downarrow & & \downarrow\alpha_1{}^* \\ \pi_{13}(PL/O_{(3)}) & \approx & \pi^s_{13(3)} \end{array}$$

which commutes by ch. 3.

But $\alpha_1{}^* : \pi^s_{10(3)} \to \pi^s_{13(3)}$ is an isomorphism [TODA], and hence $\alpha_1{}^* : \pi_{10}(PL/O_{(3)}) \to \pi_{13}(PL/O_{(3)})$ is also an isomorphism. Therefore $PL/O_{(3)}$ cannot have the homotopy type of $K(Z_3,10) \times K(Z_3,13)$ in dimensions ≤ 13, and so $k \neq 0$.

4.3. On the method. Technical lemmas.

Let $f : Y \to X$ be a map and suppose $f_* : \pi_i(Y) \to \pi_i(X)$ is an isomorphism for $i < n$ an surjective for $i = n$. If for $i \in I$, $h_i : S^n \to Y$ represent elements in $\mathrm{Ker}[f_*:\pi_n(Y) \to \pi_n(X)]$, we may form the Puppe sequence

$$\bigvee_i S^{n_i} \xrightarrow{h} Y \longrightarrow C_h ,$$

and factor f through C_h (since $f \circ h \simeq 0$). Call the factor $\tilde{f} : C_h \to X$. Then

$$\tilde{f}_* : \pi_i(C_h) \xrightarrow{\approx} \pi_i(X) \quad \text{for} \quad i < n$$

and $$\tilde{f}_* : \pi_n(C_h) \to \pi_n(X) \quad \text{is surjective,}$$

and a "better approximation" than f_*. If $\pi_n(Y)$ is finitely generated, we may continue to make $\tilde{f}_* : \pi_n(C_h) \to \pi_n(X)$ an isomorphism.

So, if X is $(n-1)$-connected and each $\pi_i(X)$ finitely generated, we start with Y a wedge of n-spheres and the components of f representing a set of generators in $\pi_n(X)$. The construction above gives us a map $\tilde{f} : C_h \to X$ inducing isomorphism

$$\tilde{f}_* : \pi_i(C_h) \to \pi_i(X) \qquad i \leq n+1$$

Then we add a set of generators of $\pi_{n+1}(X)/\tilde{f}_*\pi_{n+1}(C_h)$ to obtain a new map $f_1 : Y_1 \to X$ which induces isomorphisms on π_i, $i \leq n$ and a surjection on π_{n+1}

Proceeding this way, we may (at least theoretically) construct a CW-approximation to X.

The problem is, on each stage to calculate $\pi_i(C_h)$, $i \geq n$, and the induced map $\tilde{f}_*: \pi_*(C_h) \to \pi_*(X)$.

Now we introduce some notation:

If $h : S^n \to Y$ is a map, we will write $Y \cup_h e^{n+1}$ for C_h, and use the following notation for the Puppe sequence:

$$S^n \xrightarrow{h} Y \xrightarrow{\iota} Y \cup_h e^{n+1} \xrightarrow{q} S^{n+1} \xrightarrow{Sh} SY \to \cdots$$

If $Y = S^n$ and h is of degree m, we will write

$$Y \cup_h e^{n+1} = S^n/m.$$

To calculate the homotopy of $Y \cup_h e^{n+1}$, we shall use the following lemma:

Lemma A. Suppose Y is k-connected, and $n \geq k \geq 2$. Then we have the exact sequences

$$\text{(i)}\quad \ldots \to \pi_i(S^n) \xrightarrow{h_*} \pi_i(Y) \xrightarrow{\iota_*} \pi_i(Y \cup_h e^{n+1}) \xrightarrow{\delta} \pi_{i-1}(S^n) \xrightarrow{h_*} \pi_i(Y) \to \ldots$$

for $i \leq k+n-1$, and

$$\text{(ii)}\quad \ldots \to \pi_i(S^n) \xrightarrow{h_*} \pi_i(Y) \xrightarrow{\iota_*} \pi_i(Y \cup_h e^{n+1}) \xrightarrow{q_*} \pi_i(S^{n+1}) \xrightarrow{(Sh)_*} \pi_i(SY) \ldots$$

for $i \leq 2k+1$.

Remark: This lemma is far from the best possible, but it gives an easy formulation of all we need in the following.

Proof. (i) follows from the long, exact homotopy sequence of the pair (Y, S^n) and the homotopy excision theorem.

(ii) follows from (i) and the suspension theorem and the commutativity of the following diagram:

$$\begin{array}{ccc} \pi_i(Y, S^n) & \longrightarrow & \pi_i(Y \cup_h e^{n+1}) \\ \delta \downarrow & & \downarrow q_* \\ \pi_{i-1}(S^n) & \xrightarrow{S} & \pi_i(S^{n+1}) \end{array}$$

The extension $Y \cup_h e^{n+1} \xrightarrow{\tilde{f}} X$ may be done in different ways, and hence we may vary the induced map $f_*\colon \pi_*(Y \cup_h e^{n+1}) \to \pi_*(X)$. The possible variations are determined by Lemma B and lemma C below.

Lemma B. Suppose X is an H-space. Then $\pi_{n+1}(X)$ operates on $[Y \cup_h e^{n+1}, X]$, and the operation has the following properties:

(i) If $\alpha, \beta \in [Y \cup_h e^{n+1}, X]$, then
$\iota^* \alpha = \iota^* \beta \in [Y, X]$ if and only if there exists $x \in \pi_{n+1}(X)$ such that $\alpha = x\beta$.

(ii) For the induced maps in homotopy we have
$(x\beta)_* u = \beta_* u + (x \circ q)_* u \in \pi_*(X)$
(for all $u \in \pi_*(Y \cup_h e^{n+1})$).

Proof: From the Puppe sequence

$$S^n \xrightarrow{h} Y \xrightarrow{\iota} Y \cup e^{n+1} \xrightarrow{q} S^{n+1}$$

we get, since X is an H-space, an exact sequence of groups

$$\pi_{n+1}(X) \xrightarrow{q^*} [Y \cup_h e^{n+1}, X] \xrightarrow{\iota^*} [Y, X] \to \pi_n(X).$$

Let $x \in \pi_{n+1}$ and $\alpha \in [Y \cup_h e^{n+1}, X]$. Then we define $x\alpha = (q^* x) \cdot \alpha \in [Y \cup_h e^{n+1}, X]$. Then property (i) follows from exactness.

To prove property (ii) we look at the map

$$[Y \cup_h e^{n+1}, X] \to \mathrm{Hom}(\pi_*(Y \cup_h e^{n+1}), \pi_*(X))$$
$$\alpha \longmapsto \alpha_* .$$

Both these sets are actually groups, and since the group structures may be defined using the H-space structure on X, the map is a homomorphism. But then we have,

$$(x \cdot \beta)_* = (q^* x \cdot \beta)_* = (q^* x)_* + \beta_* = (x \circ q)_* + \beta_*.$$

So Lemma B gives us the indeterminacy of $\mathrm{Im}[\tilde{f}_*: \pi_*(Y \cup_h e^{n+1}) \to \pi_*(X)]$, but it does not help us very much in the actual calculation of the induced map.
To this end we need Lemma C.

Suppose Y is k-connected, $n > k$ and $j \leq 2k+1$. In the following diagram:

$$\begin{array}{ccccc} \pi_j(Y \cup_h e^{n+1}) & \xrightarrow{q_*} & \pi_j(S^{n+1}) & \xrightarrow{(Sh)_*} & \pi_j(SY) \\ & & S\uparrow & & \uparrow S \\ & & \pi_{j-1}(S^n) & \longrightarrow & \pi_{j-1}(Y) \end{array}$$

the top row is exact (Lemma A), and the vertical homomorphisms are isomorphisms (the suspension theorem).

Let $[\alpha] \in \pi_j(Y \cup_h e^{n+1})$. Then $[q \circ \alpha] = q_*[\alpha] = S[\alpha_1]$ where $[\alpha_1] \in \pi_{j-1}(S^n)$ and $h_*[\alpha_1] = 0$; that is, $h \circ \alpha_1 \simeq 0$.

If now $[h] \in \mathrm{Ker}[f_*: \pi_n(Y) \to \pi_n(X)]$, we can form the Toda bracket

$$\{f,h,\alpha_1\} \subset \pi_j(X).$$

<u>Lemma C.</u>

$$\{f,h,\alpha_1\} = \{\tilde{f}_*[x] \mid \tilde{f}: Y \cup_h e^{n+1} \to X \text{ extends } f, \text{ and } q_*[\alpha] = S[\alpha_1]\}.$$

<u>Proof</u>. [TODA], prop 1,7.

4.4. Generators in $\pi_*(PL/O_{(2)}), * \leq 10.$

In this range we have split exact sequences

$$0 \to bP_{n+1(2)} \to \pi_n(PL/O_{(2)}) \to \pi^s_{n(2)}/\mathrm{im}\, J_{(2)} \to 0,$$

and we know that $bP_{8(2)} \approx \mathbb{Z}_4$, $bP_{10(2)} \approx \mathbb{Z}_2$ and

$bP_{9(2)} \approx bP_{10(2)} \approx 0$ [15].

Let now $\eta \in \pi^s_{1(2)}$ be the generator (the stable Hopf map). To study the localized J-homomorphism, we look at the following diagram, which commutes

$$\left.\begin{array}{lcccl}
Z_{(2)} \approx \pi_7\,(SO)_{(2)} & \xrightarrow{J^7} & \pi^s_{7(2)} \approx Z_{16}\,, & & \sigma \\
\quad\quad\downarrow \eta^* & & \downarrow \eta^* & & \\
\mathbb{Z}_2 \approx \pi_8\,(SO)_{(2)} & \xrightarrow{J^8} & \pi^s_{8(2)} \approx Z_2 \oplus Z_2, & & \bar{\nu},\ \varepsilon \\
\quad\quad\downarrow \eta^* & & \downarrow \eta^* & & \\
\mathbb{Z}_2 \approx \pi_9\,(SO)_{(2)} & \xrightarrow{J^9} & \pi^s_{9(2)} \approx Z_2 \oplus Z_2 \oplus Z_2, & & \nu^3, \mu, \eta^*(\varepsilon) \\
\quad\quad\downarrow \eta^* & & \downarrow \eta^* & & \\
0 \approx \pi_{10}(SO)_{(2)} & \xrightarrow{J^{10}} & \pi^s_{10(2)} \approx Z_2\,, & & \eta^*(\mu)
\end{array}\right\} \begin{array}{l}\text{genera-}\\ \text{tors}\\ \text{with}\\ \text{notation}\\ \text{as in}\\ \text{[TODA].}\end{array}$$

Furthermore, as shown in [TODA], we have the relations

$$\eta^*(\sigma) = \bar{\nu} + \varepsilon,\ \eta^*(\bar{\nu}) = \nu^3,\ \eta^*(\nu^3) = 0,\ \eta^*(\eta^*(\varepsilon)) = 0.$$

$J^7 : \pi_7(SO)_{(2)} \to \pi^s_{7(2)}$ is surjective [16], and since J^8 and J^9 are imbeddings onto a direct summand [1], it follows that

$\operatorname{Im} J^8$ = the subgroup generated by $\bar{\nu} + \varepsilon \in \pi^s_{8(2)}$

$\operatorname{Im} J^9$ = the subgroup generated by $\nu^3 + \eta^*(\varepsilon) \in \pi^s_{9(2)}$.

Then we get coker $J_{(2)}$:

$\pi^s_{7(2)}/\operatorname{Im} J_{(2)} \approx 0$

$\pi^s_{8(2)}/\operatorname{Im} J_{(2)} \approx Z_2$, generated by $[\varepsilon]$ (= the coset of ε)

$\pi^s_{9(2)}/\operatorname{Im} J_{(2)} \approx Z_2 + Z_2$, generated by $[\mu]$ and $[\eta^*(\varepsilon)]$

$\pi^s_{10(2)}/\operatorname{Im} J_{(2)} \approx Z_2$, generated by $[\eta^*(\mu)]$

η^* will also induce a map between the cokernels, and - calling this also η^* - we have

$$\eta^* [\varepsilon] = [\eta^*(\varepsilon)] \neq 0$$

$$\eta^* [\mu] = [\eta^*(\mu)] \neq 0$$

$$\eta^* [\eta^*(\varepsilon)] = 0.$$

Now consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & bP_{8(2)} & \longrightarrow & \pi_7(PL/O_{(2)}) & \longrightarrow & 0 & \longrightarrow & 0 \\
& \downarrow \eta^* & & \downarrow \eta^* & & \downarrow \eta^* & & \\
0 \longrightarrow & bP_{9(2)} & \longrightarrow & \pi_8(PL/O_{(2)}) & \xrightarrow{\approx} & \pi^s_{8(2)}/\operatorname{Im} J_{(2)} & \longrightarrow & 0 \\
& \downarrow \eta^* & & \downarrow \eta^* & & \downarrow \eta^* & & \\
0 \longrightarrow & bP_{10(2)} & \longrightarrow & \pi_9(PL/O_{(2)}) & \xrightarrow{\approx} & \pi^s_{9(2)}/\operatorname{Im} J_{(2)} & \longrightarrow & 0 \\
& \downarrow \eta^* & & \downarrow \eta^* & & \downarrow \eta^* & & \\
0 \longrightarrow & bP_{11(2)} & \longrightarrow & \pi_{10}(PL/O_{(2)}) & \longrightarrow & \pi^s_{10(2)}/\operatorname{Im} J_2 & \longrightarrow & 0
\end{array}$$

From this we see, using the results above, that we can choose the following as generators in $\pi_*(PL/O_{(2)})$, $* \leq 10$

$\pi_7(PL/O_{(2)})$, A generator β_7 for $bP_{8(2)} \approx Z_4$

$\pi_8(PL/O_{(2)})$, A generator $[\varepsilon]$ corresponding to $[\varepsilon] \in \pi^S_{8(2)}/\mathrm{Im}\, J_{(2)}$

$\pi_9(PL/O_{(2)})$, $\eta^*[\varepsilon]$, an element which is mapped on $[\mu]$ in $\pi^S_{9(2)}/\mathrm{Im}\, J_{(2)}$ - we call this element too $[\mu]$ -, and a generator β_9 for $bP_{10(2)} \approx Z_2$.

$\pi_{10}(Pl/O_{(2)})$, $\eta^*[\mu]$.

With these results, we are ready to start the program sketched in 4.3.

4.5. A CW-approximation of $PL/O_{(2)}$.

We start with the mapping $\beta_7 : S^7 \to PL/O_{(2)}$. (We will use the same name on a map and its homotopy class. Particularly, $S^n \xrightarrow{m} S^n$ will mean a map of degree m.)

$\beta_{7*} : \pi_7(S^7) \to \pi_7(PL/O_{(2)})$ is surjective, and - since $\pi_8(S^7)$ and $\pi_9(S^7)$ are generated by η and η^2, and $\beta_7 \circ \eta \simeq 0$ - β_{7*} is the zero map on π_8 and π_9.

Now we kill $\mathrm{Ker}\, [\beta_{7*} : \pi_7(S^7) \to \pi_7(PL/O_{(2)})]$ by forming $S^7/4$ and taking an extension $\tilde{\beta}_7 : S^7/4 \to PL/O_{(2)}$ of β_7. By Lemma A (which we from now on will not refer to each time we use it) we get exact sequence

$$\pi_7(S^7) \xrightarrow{4} \pi_7(S^7) \longrightarrow \pi_7(S^7/4) \longrightarrow \pi_6(S^7) \approx 0 ,$$

showing that $\pi_7(S^7/4) \approx \mathbb{Z}_4$, and it is obvious that β_7 is an isomorphism on π_7.

Then we have the exact sequence

$$\pi_8(S^7) \xrightarrow{4} \pi_8(S^7) \longrightarrow \pi_8(S^7/4) \longrightarrow 0,$$

and since $\pi_8(S^7) \approx \mathbb{Z}_2$, generated by η, $\pi_8(S^7/4) \approx \mathbb{Z}_2$, generated by $S^8 \xrightarrow{\eta} S^7 \xrightarrow{\iota} S^7/4$. But, since $\eta^*[\beta_7] = 0$, $\tilde{\beta}_{7*} : \pi_8(S^7/4) \longrightarrow \pi_8(PL/O_{(2)})$ must be zero.

$\pi_9(S^7/4)$ we find in the exact sequence

$$\pi_9(S^7) \xrightarrow{4} \pi_9(S^7) \longrightarrow \pi_9(S^7/4) \longrightarrow \pi_9(S^8) \longrightarrow \pi_9(S^8) \longrightarrow ..$$

$\pi_9(S^7) \approx \mathbb{Z}_2$, generated by η^2, so we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_9(S^7) & \longrightarrow & \pi_9(S^7/4) & \longrightarrow & \pi_9(S^8) & \longrightarrow & 0 \\ & & \wr\wr & & & & \wr\wr & & \\ & & Z_2(\eta) & & & & Z_2(\eta) & & \end{array}$$

Thus $\pi_9(S^7/4)$ must be a group of order 4, i.e. $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

To determine which, we look at the diagram of Puppe sequences (n arbitrary ≥ 2)

$$\begin{array}{ccccccccccc} S^n & \xrightarrow{4} & S^n & \longrightarrow & S^n/4 & \longrightarrow & S^{n+1} & \xrightarrow{4} & S^{n+1} & \longrightarrow & .. \\ {\scriptstyle 2}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle 2} & & \| & & \\ S^n & \xrightarrow{2} & S^n & \longrightarrow & S^n/2 & \longrightarrow & S^{n+1} & \xrightarrow{2} & S^{n+1} & \longrightarrow & .. \end{array}$$

This induces in homotopy a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{n+2}(S^n) & \longrightarrow & \pi_{n+2}(S^n/4) & \longrightarrow & \pi_{n+2}(S^{n+1}) & \longrightarrow & 0 \\ & & = \downarrow & & \downarrow & & \downarrow 0 & & \\ 0 & \longrightarrow & \pi_{n+2}(S^n) & \longrightarrow & \pi_{n+2}(S^n/2) & \longrightarrow & \pi_{n+2}(S^{n+1}) & \longrightarrow & 0 \end{array}$$

with exact rows (by an argument similar to that for $n = 7$ above). Both the groups in the middle must be of order 4 and it is easy to see that this diagram can only exist with $\pi_{n+2}(S^n/4) \approx Z_2 \oplus Z_2$.

Hence $\pi_9(S^{7/4}) \approx Z_2 \oplus Z_2$ - one summand generated by $S^9 \xrightarrow{\eta^2} S^7 \xrightarrow{\iota} S^7/4$, the other generated by a map $\alpha : S^9 \to S^7/4$ such that $q \circ \alpha \simeq \eta : S^9 \xrightarrow{\alpha} S^7/4 \xrightarrow{q} S^8$.

Now we want to calculate the induced homomorphism $\tilde{\beta}_{7*} : \pi_9(S^7/4) \to \pi_9(PL/O_{(2)})$.

First we see that $\tilde{\beta}_{7*}[\iota \circ \eta^2] = [\tilde{\beta}_7 \circ \iota \circ \eta^2] = [\beta_7 \circ \eta^2] = 0$.

$\tilde{\beta}_{7*}[\alpha]$ is determined by Lemma C.

We have $S^8 \xrightarrow{\eta} S^7 \xrightarrow{4} S^7 \xrightarrow{\beta_7} PL/O_{(2)}$ with $4\eta \simeq 0$ and $4\beta_7 \simeq 0$, and we have to calculate $\{\beta_7, 4, \eta\} \subset \pi_9(PL/O_{(2)})$.

Recall that the element $[\mu] \in \pi_9(PL/O_{(2)}$ was well-defined only mod bP_{10}.

Lemma. We may choose $[\mu]$ such that

$$\{\beta_7, 4, \eta\} = [\mu] + ([\eta\varepsilon]) \subset \pi_9(PL/O_{(2)})$$

Proof. The indeterminacy is

$$\pi_9(PL/O_{(2)})\circ\eta + \beta_7 \circ \pi_9(S^7) = ([\eta\varepsilon]).$$

So we have to show that $\{\beta_7, 4, \eta\}$ contains an element which is mapped onto $[\mu] \in \pi^s_{9(2)}/\mathrm{Im}\, J_{(2)}$.

Using the natural isomorphism $\pi_{n+1}(X) \approx \pi_n(\Omega X)$, we get a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \pi_8(BO_{(2)}) \arrow[r] & \pi_8(BPL_{(2)}) \arrow[r] & \pi_7(PL/O_{(2)}) \arrow[r] & 0 \\
& \pi_7(\Omega BO_{(2)}) \arrow[r] \arrow[u, phantom, "\wr\wr"] & \pi_7(\Omega BPL_{(2)}) \arrow[u, phantom, "\wr\wr"] \arrow[ur, "h_*"'] & &
\end{tikzcd}$$

The upper line is Hirsch-Mazur's exact sequence, and $h : \Omega BPL_{(2)} \to PL/O_{(2)}$ is the fibre of $PL/O_{(2)} \to BO_{(2)}$. In this diagram $\pi_8(BO_{(2)}) \approx \pi_8(BO)_{(2)} \approx Z_{(2)}$, and $\pi_8(BPL_{(2)}) \approx \pi_8(BPL)_{(2)} \approx Z_{(2)} \oplus Z_4$ (Brumfiel [4]), and therefore the Z_4-summand must map isomorphically onto $\pi_7(PL/O_{(2)})$. Hence we can find a map $b_7 : S^7 \to \Omega BPL_{(2)}$ such that $h \circ b_7 \simeq \beta_7$.

Then $h \circ \{b_7, 4, \eta\} \subset \{\beta_7, 4, \eta\}$.

Furthermore, if p is the natural map $\Omega BPL_{(2)} \to \Omega BF_{(2)}$, we have

$$p \circ \{b_7, 4, \eta\} \subset \{p \circ b_7, 4, \eta\},$$

In the exact homotopy sequence of the weak fibration $\Omega BL_{(2)} \to \Omega BF_{(2)} \to F/PL_{(2)}$ (Localization preserves weak fibrations) we find the short exact part

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_8(F/PL_{(2)}) & \longrightarrow & \pi_7(\Omega BPL_{(2)}) & \xrightarrow{p_*} & \pi_7(\Omega BF_{(2)}) & \longrightarrow & 0 \\
 & & \wr\wr & & \wr\wr & & \wr\wr & & \\
 & & & & & & \pi^s_{7(2)} & & \\
 & & & & & & \wr\wr & & \\
0 & \longrightarrow & Z_{(2)} & \longrightarrow & Z_{(2)} \oplus Z_4 & \longrightarrow & Z_{16} & \longrightarrow & 0
\end{array}$$

The Z_4-summand must map monomorphically into Z_{16}, such that with the identification $\pi_7(\Omega BF_{(2)}) = \pi^s_{7(2)}$ (which commutes with Toda brackets) $p \circ b_7 \simeq (4a)\sigma$, where $\sigma \in \pi^s_{7(2)}$ is the generator, and $a = 1$ or 3. But from [TODA] we know that

$$\{4\sigma, 4, \eta\} \supseteq \{8\sigma, 2, \eta\} \ni \mu ,$$

and hence also

$\{p \circ b_7, 4, \eta\} \ni \mu$, considered as an element of $\pi_9(\Omega BF_{(2)}) \approx \pi^s_{9(2)}$. The indeterminacy here is

$$\pi^s_{8(2)} \circ \eta + (p \circ b_7) \circ \pi_9(S^2) = (\nu^3, \eta \circ \varepsilon)$$

which does not contain μ. Therefore $\mu \in p \circ \{b_7, 4, \eta\}$, and the Lemma follows from the diagram

$$\begin{tikzcd}
 & \pi_9(\Omega BPL_{(2)}) \arrow[r, "p_*"] \arrow[d] & \pi_9(\Omega BF_{(2)}) \approx \pi^s_{9(2)} \arrow[d] & \\
0 \arrow[r] & bP_{10(2)} \arrow[r] & \pi_9(PL/O_{(2)}) \arrow[r] & \pi^s_{9(2)}/\mathrm{Im}\, J_{(2)} \arrow[r] & 0
\end{tikzcd}$$

From this and Lemma C we now get

<u>Corollary 1</u>. We can choose the extension $\tilde{\beta}_7 : S^7/4 \to PL/O_{(2)}$ such that

$$\tilde{\beta}_{7*}[\alpha] = [\mu] \in \pi_9(PL/O_{(2)})$$

<u>Corollary 2</u>. With this choice (Corollary 1)

$$\tilde{\beta}_{7*} : \pi_{10}(S^7/4) \to \pi_{10}(PL/O_{(2)}) \text{ is surjective.}$$

<u>Proof</u>. $\pi_{10}(PL/O_{(2)})$ is generated by $[\mu]\circ\eta = \tilde{\beta}_{7*}[\alpha\circ\eta]$

$\pi_8(S^7/4)$ is mapped to zero by $\tilde{\beta}_{7*}$, and we have to kill this croup. This is done by attaching e^9 by the map $\iota\circ\eta : S^8 \xrightarrow{\eta} S^7 \xrightarrow{\iota} S^7/4$, which generates $\pi_8(S^7/4)$.

So let $X = S^7/4 \cup_{\iota\circ\eta} e^9$, and let $g : X \to PL/O_{(2)}$ be an extension of $\tilde{\beta}_7$. Then g_* is an isomorphism on π_7, and $\pi_8(X) = 0$.

To calculate $\pi_9(X)$, we use the exact sequence

$$\begin{tikzcd}
\pi_9(S^8) \arrow[r, "(\iota\circ\eta)_*"] \arrow[d, phantom, "\wr"] & \pi_9(S^7/4) \arrow[r, "(\iota_X)_*"] \arrow[d, phantom, "\wr"] & \pi_9(X) \arrow[r] \arrow[dr, "(q_X)_*"'] & \pi_8(S^8) \arrow[r, "(\iota\circ\eta)_*"] & \pi_8(S^7/4) \arrow[r] & .. \\
\mathbf{Z}_2 & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & & \pi_9(S^9) \approx \mathbf{Z} \arrow[u, "\approx/S"'] & &
\end{tikzcd}$$

($S^7/4 \xrightarrow{\iota_X} X \xrightarrow{q_X} S^9$ is the Puppe sequence).

Recalling the calculations above for $\pi_8(S^7/4)$ and $\pi_9(S^7/4)$, we see that we get an exact sequence

$$0 \to ([\alpha]) \to \pi_9(X) \to 2\pi_9(S^9) \to 0 \ ,$$

where $[\alpha]$ is a generator in $\pi_9(S^7/4)$. Since $\pi_9(S^9) \approx Z$, we get $\pi_9(X) \approx Z \oplus Z_2$. The Z_2-summand is generated by $\iota_X \circ \alpha : S^9 \xrightarrow{\alpha} S^7/4 \to X$, and the Z-summand is generated by an element β such that the composition $S^9 \xrightarrow{\beta} X \xrightarrow{q_X} S^9$ is of degree 2.

Thus the Z_2-summand is mapped onto $([\mu] \subset \pi_9(PL/O_{(2)})$ by g_*.

By Lemma C, $g_*([\beta]) \in \{\tilde{\beta}_7, \iota\circ\eta, 2\}$, so we must calculate this bracket.

Lemma. $\quad 0 \in \{\beta_7, \eta, 2\} \subset \pi_9(PL/O_{(2)})$

Proof. Since $\pi_7(PL/O_{(2)}) = bP_{8(2)}$, β_7 lifts to $(\Omega\ F/PL)_2 \simeq \Omega(F/PL_{(2)})$ (ch 3).

But as Sullivan has shown,

$$F/PL_{(2)} \simeq E \times \prod_{n=2}^{\infty} (K(Z_2, 4n) \times K(Z_2, 4n-2))$$

where $\pi_k(E) = 0$ for $k > 5$. Therefore β_7 lifts to $K(Z_{(2)}, 7)$, and we have the diagram

$$\begin{tikzcd} & & & K(Z_{(2)},7) \arrow[d, "h'"] \\ S^8 \arrow[r, "2"] & S^8 \arrow[r, "\eta"] & S^7 \arrow[ur, "\beta_7'"] \arrow[r, "\beta_7"] & PL/O_{(2)} \end{tikzcd}$$

Obviously $\beta_7' \circ \eta \simeq 0$, hence $\{\beta_7', \eta, 2\}$ is defined, and equal to 0 since $\pi_9(K(Z_{(2)}, 7)) = 0$.

Therefore

$$0 \in h' \cdot \{\beta_7', \eta, 2\} \subset \{h' \circ \beta_7', \eta, 2\} = \{\beta_7, \eta, 2\} ,$$

proving the lemma.

But then we also have

$$0 \in \{\beta_7, \eta, 2\} = \{\tilde{\beta}_7 \circ \iota, \eta, 2\} \subset \{\tilde{\beta}_7, \iota \circ \eta, 2\} ,$$

and by Lemma C we can choose g such that $g^*([\beta]) = 0$.

Now the Z-summand is mapped to zero, and we kill it by forming $X_1 = X \cup_\beta e^{10}$ and letting g_1 be an extension of g to X_1. Then g_{1*} has the following properties:

isomorphism on π_8

zero on π_8 . $\quad (\pi_8(X_1) = 0)$

imbedding onto a Z_2-summand on π_9

surjective on π_{10} .

Now let $\tilde{\varepsilon}: S^8/2 \to PL/O_{(2)}$ represent the generator in $\pi_8(PL/O_{(2)})$, and $\tilde{\beta}_9: S^9/2 \to PL/O_{(2)}$ the generator in $bP_{10(2)} \subset \pi_9(PL/O_{(2)})$.

Then $\pi_9(S^8/2) \approx Z_2$, generated by $\iota \circ \eta: S^9 \xrightarrow{\eta} S^8 \to S^8/2$, and

$$\tilde{\varepsilon}_*[\iota \circ \eta] = [\tilde{\varepsilon} \circ \iota \circ \eta] = [\varepsilon \circ \eta] = \eta^*[\varepsilon] \in \pi_9(PL/O_{(2)})$$

Thus, letting $Y = X_1 \vee S^8/2 \vee S^9/2$ and

$f = g_1 \vee \tilde{\varepsilon} \vee \tilde{\beta}_9 : Y \to PL/O_{(2)}$, we have

f_* is an isomorphism on π_i , $i \leq 9$

f_* surjective on π_{10} .

(If Z_1 is m-connected and Z_2 n-connected, then $\pi_i(Z_1 \vee Z_2) \approx \pi_i(Z_1) \oplus \pi_i(Z_2)$, $i < m+n$) . But, by Whitehead's theorem, the same is true in homology; that is

$$f_* : H_i(Y) \to H_i(PL/O_{(2)})$$

is an isomorphism for $i \leq 9$, and an epimorphism for $i = 10$.

Now Y is given with an explicit cell subdivision:

$$Y = ((S^7 \cup_4 e^8 \cup_\eta e^9) \cup_\beta e^{10}) \vee (S^8 \cup_2 e^9) \vee (S^9 \cup_2 S^{10}) ,$$

and it is easy to find its homology.
We get $H_7(Y) = Z_4$, $H_8(Y) \approx Z_2$, $H_9(Y) \approx Z_2 \oplus Z_2$ and $\tilde{H}_i(Y) = 0$ otherwise.

Then we have the homology of $PL/O_{(2)}$.

<u>Theorem</u>.
$$\tilde{H}_i(PL/O_{(2)}) = 0 \quad \text{for} \quad i < 7$$
$$H_7(PL/O_{(2)}) = Z_4$$
$$H_8(PL/O_{(2)}) = Z_2$$
$$H_9(PL/O_{(2)}) = Z_2 + Z_2$$
$$H_{10}(PL/O_{(2)}) = 0 .$$

If we let $h_i : \pi_i(X) \to H_i(X)$ be the Hurewicz homomorphism, we easily get the additional information.

<u>Theorem</u>. h_7 and h_8 are isomorphisms.
h_9 maps one Z_2-summand isomorphic and the other two to zero.

Passing to Z_2-cohomology, we get the following Steenrod algebra structure. (Recall that $H^*(PL/O, Z_2) \approx H^*(PL/O_{(2)}, Z_2)$)

<u>Theorem</u>.
$$H^7(PL/O, Z_2) \approx Z_2 \text{, generator } \alpha$$
$$H^8(PL/O, Z_2) \approx 2Z_2 \text{, generator } \beta, d_2\alpha$$
$$H^9(PL/O, Z_2) \approx 3Z_2 \text{, generator } \gamma, Sq^1\beta, Sq^2\alpha$$
$$H^{10}(PL/O, Z_2) \approx 2Z_2 \text{, generator } Sq^1\gamma, Sq^3\alpha .$$

(d_2 is a second order Bockstein, defined on elements x with $Sq^1 x = 0$).

<u>Proof</u>: In this range we have an isomorphism of modules over the Steenrod algebra

$$H^*(PL/O, Z_2) \approx H^*(X_1, Z_2) \oplus H^*(S^8/2, Z_2) \oplus H^*(S^9/2, Z_2).$$

The additive structure is given by the following table

	$H^k(X_1, Z_2)$	$H^k(S^8/2, Z_2)$	$H^k(S^9/2, Z_2)$
$k = 7$	$Z_2(\alpha)$	0	0
$k = 8$	Z_2	$Z_2(\beta)$	0
$k = 9$	Z_2	Z_2	$Z_2(\gamma)$
$k = 10$	Z_2	0	Z_2

$Sq^1\alpha = 0$, since α lifts to Z_4-coefficients. Then the theorem follows since Sq^1 detects a map of degree 2, d_2 detects a map of degree 4, and Sq^2 detects η. Note also the Adem relation $Sq^3 = Sq^1 Sq^2$.

Ch. 5. The first k-invariants of $PL/O_{(2)}$.

As already noted, α lifts to Z_4-coefficients. Let $\tilde{\alpha} : PL/O_{(2)} \to K(Z_4,7)$ be a lifting.

We start with

$$q = (\tilde{\alpha},\beta,\gamma) : PL/O_{(2)} \to K(Z_4,7) \times K(Z_2,8) \times K(Z_2,9) = \Pi$$

Then q is 8-connected, and - assuming that $F \to PL/O_{(2)} \xrightarrow{q} \Pi$ is a fibration - we have

$$\pi_9(F) \approx 2Z_2 \quad \text{and} \quad \pi_i(F) = 0 \quad \text{for} \quad i \leq 8.$$

Look at the Serre cohomology sequence for this fibration, with Z_2-coefficients:

$$\dots H^9(\Pi) \xrightarrow{q^*} H^9(PL/O_{(2)}) \to H^9(F) \xrightarrow{\tau} H^{10}(\Pi) \xrightarrow{q^*} H^{10}(PL/O_{(2)}) \dots$$

$$H^9(F;Z_2) \approx \mathrm{Hom}(H_9(F;Z);\ Z_2) \approx \mathrm{Hom}(\pi_9(F);\ Z_2) \approx 2Z_2\ ,$$

and we let I_1 and I_2 be two generators. (Corresponding to two linearly independent projections $\pi_9(F) \to Z_2$)

$$H^{10}(\Pi) = (Sq^2 d_2 \iota_7, Sq^3 \iota_7, Sq^2 \iota_8, Sq^1 \iota_9)$$

(ι_7 is the fundamental class of $K(Z_4,7)$ etc.) and

$$q^* c_7 = \alpha\ ,\ q^* \iota_8 = \beta\ ,\ q^* \iota_9 = \gamma\ .$$

Hence $\mathrm{Im}\ \tau = \ker[q^* : H^{10}(\Pi) \to H^{10}(PL/O_{(2)})] = (Sq^2 d_2 \iota_7, Sq^2 \iota_8)$ and we may choose $k_1 = (Sq^2 d_2 \iota_7, Sq^2 \iota_8)$ as the first k-invariant in $H^{10}(\Pi, 2Z_2)$.

The fibre of k_1 we call $E_1 \xrightarrow{p_1} \Pi$, and from general theory (f.ex. [36]), we know that q lifts to a map $q_1 : PL/O_{(2)} \to E_1$ which is 9-connected.

The fibre of q_1 we call F_1, and then $\pi_{10}(F_1) \approx Z_2$ and $\pi_i(F_1) = 0$ for $i \leq 9$.

$$\begin{array}{ccccc} & & F_1 & \longrightarrow & PL/O_{(2)} \\ & & & & \downarrow q_1 \\ K(2Z_2,9) & \longrightarrow & & & E_1 \xrightarrow{k_2} K(Z_2,11) \\ & & & & \downarrow p_1 \\ & & & & \Pi \xrightarrow[k_1]{} K(2Z_2,10) \end{array}$$

Just as for k_1, we now find the next k-invariant $k_2 \in H^{11}(E_1, \pi_{10}(F_1))$ by computing the transgressian of the fundamental class (= generator) in $H^{10}(F_1, \pi_{10}(F_1))$ in the fibration $F_1 \to PL/O_{(2)} \to E_1$.

Again we look at the Serre sequence:

$$H^{10}(E_1) \xrightarrow{q_1^*} H^{10}(PL/O_{(2)}) \longrightarrow H^{10}(F_1) \xrightarrow{\tau} H^{11}(E_1) \xrightarrow{q_1^*} H^{11}(PL/O_{(2)})$$

(Z_2-coefficients).

Since $q_1^* \circ p_1^* = q^* : H^{10}(\Pi) \to H^{10}(PL/O_{(2)})$ is surjective, q_1^* is also surjective. Hence τ is injective, and $k_2 \neq 0$.

Then it suffices to find an element $\neq 0$ in

$$\ker[q_1^*: H^{11}(E_1) \to H^{11}(PL/O_{(2)})]$$

First of all we calculate $H^{11}(E_1)$. To this end we use the Serre sequence for the fibration $E_1 \xrightarrow{p_1} \Pi \xrightarrow{k_1} K(2Z_2,10)$:

$$\begin{tikzcd}[column sep=small] H^{11}(K(2Z_2,10)) \arrow[r, "k_1^*"] & H^{11}(\Pi) \arrow[r, "p_1^*"] & H^{11}(E_1) \arrow[r, "\tau"] \arrow[dr, "\iota_1^*"'] & H^{12}(K(2Z_2,10)) \arrow[r, "k_1^*"] \arrow[d, phantom, "\wr"] & H^{12}(\Pi) \\ & & & H^{11}(K(2Z_2,9)) \arrow[ur, "\tau'"'] & \end{tikzcd}$$

where $i_1 : K(Z_2,9) \to E_1$ is the fibre of p_1.

$H^{10}(K(2Z_2,10))$ is generated by ι'_{10} and ι''_{10}, such that $k_1^*\iota'_{10} = Sq^2 d_2\iota_7$, and $k_1^*\iota''_{10} = Sq^2\iota_8$ in $H^{10}(\Pi)$.

Then $H^{12}(K(2Z_2,10))$ is generated by $Sq^2\iota'_{10}$ and $Sq^2\iota''_{10}$, and it is easy to see that we get an exact sequence

$$\begin{array}{ccccccccc} 0 \to & (Sq^4\iota_7, Sq^2Sq^1\iota_8, Sq^2\iota_9) & \xrightarrow{p_1^*} & H^{11}(E_1) & \xrightarrow{\tau} & (Sq^2\iota'_{10}) & \to 0 \\ & \cap & & & & \cap & \\ & H^{11}(\Pi) & & & & H^{12}(K(2Z_2,10)) & . \end{array}$$

Or, using the isomorphism $H^i(K(2Z_2,10)) \approx H^{i-1}(K(2Z_2,9))$, we get to the right

$$\begin{array}{ccl} & \xrightarrow{\tau} & (Sq^2\iota'_{10}) \to 0 \\ \to H^{11}(E_1) & & \quad \wr\wr \\ & \xrightarrow{i_1^*} & (Sq^2\iota'_9) \to 0 \end{array}$$

where ι'_9 corresponds to ι'_{10} and ι''_9 to ι''_{10}.

<u>Lemma</u>. The composition

$$(Sq^4\iota_7, Sq^2Sq^1\iota_8, Sq^2\iota_9) \xrightarrow{p_1^*} H^{11}(E_1) \xrightarrow{q_1^*} H^{11}(PL/O_{(2)})$$

is injective.

<u>Proof</u>. We must prove that $Sq^4\alpha, Sq^2Sq^1\beta$ and $Sq^2\gamma$ are linearly independent elements of $H^{11}(PL/O_{(2)})$, and we do this by constructing another space X and a map $f: X \to PL/O_{(2)}$ such that $f^*(Sq^4\alpha)$, $f^*(Sq^2Sq^1\beta)$ and $f^*(Sq^2\gamma)$ are independent elements in $H^{11}(X)$.

Let $\nu: S^{10} \to S^7$ represent a generator in $\pi_{10}(S^7)_{(2)} \approx Z_8$.

Then there is a diagram

$$\begin{array}{ccccccc} 0 & \to & bP_{8(2)} & \to & \pi_7(PL/O_{(2)}) & \to & 0 \\ & & \downarrow \nu^* & & \downarrow \nu^* & & \downarrow \nu^* \\ & & 0 & \longrightarrow & \pi_{10}(PL/O_{(2)}) & \to & \pi^S_{10(2)}/\mathrm{Im}\, J_{(2)}) \end{array},$$

showing that $\beta_7 \circ \nu \simeq 0$, where β_7 generates $\pi_7(PL/O_{(2)})$. Let $\overline{\beta}_7 : S^7 \cup_\nu e^{11} \to PL/O_{(2)}$ be an extension of β_7.

In the same way we get an extension

$$\overline{\beta}_q : S^9 \cup_\eta e^{11} \to PL/O_{(2)} \quad \text{of} \quad \beta_q : S^9 \to PL/O_{(2)}.$$

Recall from 4.5 that we had a mapping $\tilde{\varepsilon} : S^8/2 \to PL/O_{(2)}$ which induced an isomorphism on π_8. As usual this mapping is not unique, but we will show:

<u>Sublemma</u>: We can choose $\tilde{\varepsilon} : S^8/2 \to PL/O_{(2)}$ such that the induced homomorphism on π_{10} is zero.

<u>Proof</u>: From the Puppe sequence

$$S^8 \xrightarrow{2} S^8 \xrightarrow{\iota} S^8/2 \xrightarrow{q} S^9 \xrightarrow{2} S^9$$

and Lemma A (4.3), we get an exact sequence

$$0 \to \pi_{10}(S^8) \xrightarrow{\iota^*} \pi_{10}(S^8/2) \xrightarrow{q^*} \pi_{10}(S^9) \to 0$$

$\pi_{10}(S^8)$ is generated by η^2, and since for all choices of $\tilde{\varepsilon}$ we have $\tilde{\varepsilon}_* \iota_*[\eta^2] = \varepsilon_*[\eta^2] = 0$, it suffices to show that we can choose $\tilde{\varepsilon}$ such that $\tilde{\varepsilon}([z]) = 0$, where $[z] \in \pi_{10}(S^8/2)$ is such that the composition

$$S^{10} \xrightarrow{z} S^8/2 \xrightarrow{q} S^9 \text{ is homotopic to } \eta.$$

Suppose now that we have chosen an $\tilde{\varepsilon}$ with $\tilde{\varepsilon}_*[z] \neq 0$, i.e. $\tilde{\varepsilon}_*[z] = \eta^*[\mu] \in \pi_{10}(PL/O_{(2)})$. By Lemma B (4.3) we can replace $\tilde{\varepsilon}$ with another extension $[\mu]\tilde{\varepsilon}$ ($[\mu] \in \pi_9(PL/O_{(2)})$), and we have

$$\begin{aligned}([\mu]\tilde{\varepsilon})_*[z] &= \tilde{\varepsilon}_*[z] + (\mu\circ q)_*[z] \\ &= \eta^*[\mu] + \mu_*[\eta] = 2\eta^*[\mu] = 0.\end{aligned}$$

Hence $([\mu]\tilde{\varepsilon})_*$ is zero on π_{10}.

With this choice of $\tilde{\varepsilon}$, we can find an extension

$$e\colon S^8/2 \cup_z e^{11} \to PL/O_{(2)}.$$

Let now $X = (S^7 \cup_\nu e^{11}) \vee (S^9 \cup_\eta e^{11}) \vee (S^8/2 \cup_z e^{11})$ and $f = \bar{\beta}_7 \vee \bar{\beta}_9 \vee e : X \to PL/O_{(2)}$.

Letting the generators in $H^7(S^7 \cup_\nu e^{11})$, $H^8(S^8/2 \cup_z e^{11})$ and $H^9(S^9 \cup_\eta e^{11})$ be a,b and c resp., we have

$$\bar{\beta}_7{}^*(\alpha) = a, \bar{\beta}_9{}^*(\gamma) = c \quad \text{and} \quad e^*(\beta) = b.$$

Now Sq^2 detects η and Sq^4 detects ν, so we have $Sq^4 a \neq 0$ and $Sq^2 c \neq 0$.

To show that $Sq^2 Sq^1 b \neq 0$, consider the following diagram of Puppe sequences:

$$\begin{array}{ccccccc} S^{10} & \xrightarrow{z} & S^8/2 & \to & S^8/2 \cup_z e^{11} & \to & S^{11} \\ \| & & \downarrow q & & \downarrow \tilde{q} & & \| \\ S^{10} & \xrightarrow{\eta} & S^9 & \to & S^9 \cup_\eta e^{11} & \to & S^{11} \end{array}$$

In the diagram

$$\begin{array}{ccc} H^9(S^9 \cup_\eta e^{11}) & \underset{\approx}{\xrightarrow{\tilde{q}_*}} & H^9(S^8/2 \cup_z e^{11}) \\ \downarrow Sq^2 & & \downarrow Sq^2 \\ H^{11}(S^9 \cup_\eta e^{11}) & \underset{\tilde{q}_*}{\xrightarrow{\approx}} & H^{11}(S^8/2 \cup_z e^{11}) \end{array}$$

the horizontal maps are clearly isomorphisms, and the same is the left Sq^2. Then the right Sq^2 is also an isomorphism. But the generator in $H^9(S^8/2 \cup_z e^{11}) \approx H^9(S^8/2)$ is $Sq^1 b$, hence $Sq^2 Sq^1 b \neq 0$.

In the diagram

$$\begin{tikzcd} & & H^*(S^7 \cup_\nu e^{11}) \\ H^*(PL/O_{(2)}) \arrow[r, "f^*"] & H^*(X) \arrow[ur, "p_1"] \arrow[r, "p_2"] \arrow[dr, "p_3"] & H^*(S^8/2 \cup_z e^{11}) \\ & & H^*(S^9 \cup_\eta e^{11}) \end{tikzcd}$$

we have $p_1 \circ f^* = \bar{\beta}_7{}^*$, $p_2 \circ f^* = e^*$ and $p_3 \circ f^* = \bar{\beta}_9{}^*$

To show that $Sq^4 a$, $Sq^2 Sq^1 b$ and $Sq^2 c$ are linearly independent in $H^{11}(X)$, It then suffices to show that

$$\bar{\beta}_7{}^*(\beta) \in H^8(S^7 \cup_\nu e^{11}) \ , \ \bar{\beta}_7{}^*(\gamma) \in H^9(S^7 \cup_\nu e^{11}),$$

$$\bar{\beta}_9{}^*(\alpha) \in H^7(S^9 \cup_\eta e^{11}) \ , \ \bar{\beta}_9{}^*(\beta) \in H^8(S^9 \cup_\eta e^{11})$$

$$e^*(\alpha) \in H^7(S^8/2 \cup_z e^{11}), \text{ and } e^*(\gamma) \in H^9(S^8/2 \cup_z e^{11})$$

are all zero. But all the groups except $H^9(S^8/2 \cup_z e^{11})$ are zero, so the only thing we have to show, is that $e^*(\gamma) = 0$.

Since $H^9(S^8/2 \cup_z e^{11}) \to H^9(S^8/2)$ is an isomorphism and e is an extension of $\tilde{\varepsilon} : S^8/2 \to PL/O$, it suffices to show that $\tilde{\varepsilon}^*(\gamma) = 0$ in $H^9(S^8/2)$. But this follows from the isomorphism

$$H^9(PL/O_{(2)}) \xrightarrow{\approx} H^9(X_1 \vee S^8/2 \vee S^9/2) \quad \text{and}$$

the definition of γ in 4.5.

If we call the fibre of k_2 $E_2 \xrightarrow{p_2} E_1$, q_1 may be factored through p_2 :

$$q_1 = p_2 \circ q_2 : PL/O_{(2)} \xrightarrow{q_2} E_2 \xrightarrow{p_2} E_1 \ ,$$

and we have proved:

<u>Theorem:</u> We have the first two stages in a Postnikov resolution of $PL/O_{(2)}$:

$$\begin{array}{ccccc}
 & & PL/O_{(2)} & & \\
 & & \downarrow q_2 & & \\
K(Z_2,10) & \xrightarrow{i_2} & E_2 & & \\
 & & \downarrow p_2 & & \\
K(2Z_2,9) & \xrightarrow{i_1} & E_1 & \xrightarrow{k_2} & K(Z_2,11) \\
 & & \downarrow p_1 & & \\
\end{array}$$

$$\Pi = K(Z_4,7) \times K(Z_2,8) \times K(Z_2,9) \xrightarrow{(Sq^2 d_2 \iota_7, Sq^2 \iota_8)} K(2Z_2,10)$$

In this range we have no odd torsion except 3-torsion and 7-torsion in $\pi_*(PL/O)$, and these are "generated" by the fundamental classes

$$PL/O_{(3)} \xrightarrow{g(3)} K(Z_3,10) \quad \text{and} \quad PL/O_{(7)} \xrightarrow{g(7)} K(Z_7,7).$$

Letting $E_0 = K(Z_3,10) \times K(Z_7,10)$ we then get the first stages of a Postnikov resolution of PL/O:

$$
\begin{tikzcd}
 & PL/O \arrow[d] \arrow[dd, bend right=60, "\bar{q}"'] & & \\
 & PL/O_{(2)} \times PL/O_{(3)} \times PL/O_{(7)} \arrow[d, dashed, "q_2 \times g_{(3)} \times g_{(7)}"] & & \\
K(Z_2,10) \arrow[r, "i_2"] & E_2 \times E_0 \arrow[d, dashed, "p_2 \times id"] & & \\
K(2Z_2,9) \arrow[r, "i_1"] & E_1 \times E_0 \arrow[r, "pr_1"] \arrow[d, dashed, "p_2 \times id"] & E_1 \arrow[r, "k_2"] & K(Z_2,11) \\
 & \Pi \times E_0 \arrow[r, "pr"] & \Pi \arrow[r, "k_1"] & K(2Z_2,11)
\end{tikzcd}
$$

$\bar{q}$ is an 11-equivalence, and we get

<u>Corollary</u>. If M is a manifold of dimension ≤ 10 and with $\Gamma(M) \neq \emptyset$, then

$$\Gamma(M) \approx [M,E_2] \oplus H^7(M,Z_7) \oplus H^{10}(M,Z_3).$$

In particular, if dim $M \leq 8$, then

$$\Gamma(M) \approx H^7(M,Z_{28}) \oplus H^8(M,Z_2)$$

For example, we now easily get the following result for compact, connected 7-manifolds:

<u>Theorem</u>. Let M be a compact, connected 7-manifold.
If M is orientable, then $\Gamma(M) \approx Z_{28}$.
If M is nonorientable, then $\Gamma(M) \approx Z_2$.

<u>Proof</u>: In the first case, $H_7(M,Z) \approx Z$, and $H_6(M,Z)$ is torsion-free. Then

$$\Gamma(M) \approx H^7(M,Z_{28}) \approx \mathrm{Hom}(Z,Z_{28}) \oplus \mathrm{Ext}(\text{free},\ Z_{28}) \approx Z_{28}\ .$$

In the second case, $H_7(M,Z) = 0$ and $H_6(M,Z)$ has Z_2 as torsion subgroup.

Hence

$$\Gamma(M) \approx \mathrm{Hom}(0,Z_{28}) \oplus \mathrm{Ext}(Z_2 \oplus \text{free},\ Z_{28}) \approx Z_2$$

For projective spaces of dimension ≤ 8, we have

<u>Theorem</u>.

$$\Gamma(RP^7) \approx Z_{28}$$

$$\Gamma(RP^8) \approx Z_2 \oplus Z_2$$

$$\Gamma(CP^4) \approx Z_2$$

$$\Gamma(QP^2) \approx Z_2\ .$$

To illustrate calculations in higher dimensions, we now compute $\Gamma(RP^9)$. For any 9-manifold we have $\Gamma(M^9) \approx [M^9,E_1]$, so we must calculate $\Gamma(RP^9) \approx [RP^9,E_1]$.

Consider the exact sequence

$$\begin{array}{ccccccccc} [RP^9,\Omega\Pi] & \xrightarrow{(\Omega k_1)_*} & [RP^9,K(2Z_2,9)] & \to & [RP^9,E_1] & \to & [RP^9,E_1] & \to & [RP^9,K(2Z_2,10)] \\ \| & & \| & & & & \| & & \| \end{array}$$

$$\left.\begin{array}{ccc} H^6(RP^9,Z_4) & \xrightarrow{Sq^2 d_2} & H^9(RP^9,Z_2) \\ \oplus & & \oplus \\ H^7(RP^9,Z_2) & \xrightarrow{Sq^2} & H^9(RP^9,Z_2) \\ \oplus & & \oplus \\ H^8(RP^9,Z_2) & \longrightarrow & 0 \end{array}\right\} \to [RP^9,E_1] \to \left\{\begin{array}{c} H^7(RP^9,Z_4) \\ \oplus \\ H^8(RP^9,Z_2) \\ \oplus \\ H^9(RP^9,Z_2) \end{array}\right\} \to 0$$

Since d_2 is the Bockstein homomorphism of the exact sequence $0 \to Z_2 \to Z_8 \to Z_4 \to 0$, we can study d_2 by studying the homomorphism

$$H^*(RP^9,Z_8) \to H^*(RP^9,Z_4).$$

We need to calculate $d_2 : H^6(RP^9,Z_4) \to H^7(RP^9,Z_2)$, and look at the diagram

$$\begin{array}{ccccccc}
0 \to & Ext(H_5(RP^9,Z),Z_8) & \xrightarrow{\approx} & H^6(RP^9,Z_8) & \to & Hom(H_6(RP^9,Z),Z_8) & = 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & Ext(H_5(RP^9,Z),Z_4) & \xrightarrow{\approx} & H^6(RP^9,Z_4) & \to & Hom(H_6(RP^9,Z),Z_4) & = 0
\end{array}$$

Since $Ext(Z_2,Z_8) \to Ext(Z_2,Z_4)$ is an isomorphism, it follows that $d_2 : H^6(RP^9,Z_4) \to H^7(RP^9,Z_4)$ is zero.

It remains to calculate $Sq^2:H^7(RP^9,Z_2) \to H^9(RP^9,Z_2)$. $H^7(RP^9,Z_2)$ is generated by t^7, where $t \in H^1(RP^9,Z_2)$, and for one-dimensional classes we have [23] $Sq^i(t^k) = \binom{k}{i}t^{i+k}$. Therefore $Sq^2t^7 = t^9$, and hence $Sq^2 : H^7(RP^9,Z_2) \to H^9(RP^9,Z_2)$ an isomorphism.

Finally this gives us an exact sequence

$$0 \to Z_2 \to [RP^9,E_1] \to Z_2 \oplus Z_2 \oplus Z_2 \to 0 ,$$

and

$$\Gamma(RP^9) \approx Z_2 \oplus Z_2 \oplus G , \text{ where } G \text{ is a group of order 4.}$$

Ch. 6. On $H^*(BSPL)$ and $H^*(BPL)$.

Remark. $H^*(BSPL,Z_2)$ and $H^*(BPL,Z_2)$ has been completely calculated by Madsen, May and Milgram [19]. The following may be regarded as an application of the results above, rather than an announcement of new results.

Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be a fibration, and consider the pullback diagram

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \iota \\ p^*E & \xrightarrow{\tilde{p}} & E \\ \bar{p}\downarrow & & \downarrow p \\ E & \xrightarrow{p} & B \end{array}$$

We have a diagram of long, exact cohomology sequences (arbitrary coefficients)

$$\begin{array}{ccccccccc} \ldots \longrightarrow H^i(B) & \xrightarrow{p^*} & H^i(E) & \longrightarrow & H^{i+1}(B,E) & \longrightarrow & H^{i+1}(B) & \longrightarrow & \ldots \\ \downarrow p^* & & \downarrow \bar{p}^* & & \downarrow (\bar{p},p)^* & & \downarrow p^* & & \\ \ldots \longrightarrow H^i(E) & \xrightarrow{\tilde{p}^*} & H^i(p^*E) & \longrightarrow & H^{i+1}(E,p^*E) & \longrightarrow & H^{i+1}(E) & \longrightarrow & \ldots \end{array}$$

Suppose now that B is simply connected, $\tilde{H}_i(F,Z) = 0$ for $i < a$ and $H_i(B,E;Z) = 0$ for $i < b$. Then an easy spectral sequence argument shows that $H^i(B,E) \to H^i(E,p^*E)$ is an isomorphism for all $i < a+b$. Hence, for $i < a+b-1$ we may write the diagram above this way:

$$
\begin{tikzcd}[column sep=small]
H^i(B) \arrow[r, "p^*"] \arrow[d, "p^*"] & H^i(E) \arrow[d, "\overline{p}^*"] \arrow[dr] & & H^{i+1}(B) \arrow[r, "p^*"] \arrow[d, "p^*"] & H^{i+1}(E) \arrow[d, "\overline{p}^*"] \\
& & H^{i+1}(B,E) \arrow[ur] \arrow[dr] & & \\
H^i(E) \arrow[r, "\tilde{p}^*"] & H^i(p^*E) \arrow[ur, "\tau_o"'] & & H^{i+1}(E) \arrow[r] & H^{i+1}(p^*E)
\end{tikzcd}
$$

(τ_o is the relative transgression [36]).

The map $H^{i+1}(B,E) \to H^{i+1}(E)$ in this diagram is the composition of two maps in the cohomology sequence of the pair (B,E) - hence zero. Also - observing that $\tilde{p} = \overline{p}\circ T$, where $T(x,y) = (y,x)$ on $p^*E \subset E \times E$ - we may show that the map $H^i(E) \to H^i(B,E)$ is essentially the composition of two maps in the cohomology sequence of the pair (E,p^*E).

Consequently, this diagram splits in short exact sequences

$$
\left.\begin{array}{ll}
 & 0 \to H^{i+1}(B,E) \to H^{i+1}(B) \to H^{i+1}(E) \to 0 \\
\text{and} & 0 \to H^i(E) \to H^i(p^*E) \to H^{i+1}(B,E) \to 0
\end{array}\right\} \quad i < a+b-1
$$

These may again be combined to an exact sequence

$$
(*) \qquad 0 \to H^i(E) \xrightarrow{\overline{p}^*} H^i(p^*E) \to H^{i+1}(B) \xrightarrow{p^*} H^{i+1}(E) \to 0
$$

Suppose further that the fibration is principal. Then there exists a homotopy equivalence $\varphi : p^*E \to E\times F$ uch that the diagram

$$
\begin{tikzcd}
p^*E \arrow[rr, "\varphi"] \arrow[dr, "\overline{p}"'] & & E\times F \arrow[dl, "pr"] \\
& E &
\end{tikzcd}
$$

commutes.

In this case we may write $(*)$ in the following way

$$
0 \to H^i(E) \xrightarrow{pr^*} H^i(E\times F) \to H^{i+1}(B) \to H^{i+1}(E) \to 0.
$$

Consequently, there is a short exact sequence

$$0 \to \operatorname{coker}[pr^*: H^i(E) \to H^i(E\times F)] \to H^{i+1}(B) \to H^{i+1}(E) \to 0.$$

With field coefficients, this sequence splits, and hence determines the additive structure of $H^*(B)$ from that on $H^*(E)$ and $H^*(F)$.

Now we apply this to the principal fibration $PL/O \to BSO \to BSPL$ and Z_2-coefficients.

From the Künneth formula, we get

$$\operatorname{coker}[H^k(BSO) \xrightarrow{pr^*} H^k(BSO\times PL/O)] \approx \sum_{i=0}^{k-1} H^i(BSO) \otimes H^{k-i}(PL/O),$$

so our results on $H^*(PL/O)$ give:

<u>Theorem</u>.

$$H^i(BSPL) \approx H^i(BSO) \qquad i \leq 7$$
$$H^8(BSPL) \approx H^8(BSO) \oplus Z_2$$
$$H^9(BSPL) \approx H^9(BSO) \oplus 2Z_2$$
$$H^{10}(BSPL) \approx H^{10}(BSO) \oplus 4Z_2$$
$$H^{11}(BSPL) \approx H^{11}(BSO) \oplus 5Z_2 .$$

To obtain $H^*(BPL)$ in the same range, we use a theorem of Browder, Liulevicius and Peterson [5] :

There exists a Hopf algebra over the Steenrod algebra C^*, and isomorphisms

(i) $H^*(BPL) \approx H^*(BO) \otimes C^*$ and

(ii) $H^*(BSPL) \approx H^*(BSO) \otimes C^*$.

The theorem above and (ii) now give C^i, $i \leq 11$:

<u>Lemma</u>: $$C^i = 0 \;,\quad 0 < i < 8 \;,\quad {}^{o} = Z_2$$
$$C^8 = Z_2$$
$$C^9 = 2Z_2$$
$$C^{10} = 3Z_2$$
$$C^{11} = 2Z_2$$

Hence from the isomorphism (i)

<u>Corollary</u>: $$H^i\,(BPL) \approx H^i\,(BO), \quad i \leq 7$$
$$H^8\,(BPL) \approx H^8\,(BO) \oplus Z_2$$
$$H^9\,(BPL) \approx H^9\,(BO) \oplus 3Z_2$$
$$H^{10}(BPL) \approx H^{10}(BO) \oplus 7Z_2$$
$$H^{11}(BPL) \approx H^{11}(BO) \oplus 12Z_2 \;.$$

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